



Research paper

Dual generalized inverses and their use in solving systems of linear dual equations

Firdaus E. Udwardia

Civil and Environmental Engineering, Aerospace and Mechanical Engineering, Mathematics, and Information and Operations Management, Olin Hall, University of Southern California, Los Angeles, CA 90089-1453, United States

ARTICLE INFO

Article history:

Received 29 September 2020

Revised 22 October 2020

Accepted 22 October 2020

Available online 27 November 2020

Keywords:

Properties of generalized inverses

Uniqueness of Moore-Penrose dual generalized inverse

Explicit solution of system of dual equations

Necessary and sufficient conditions for existence of {1}-dual generalized inverse

Dual analog of least-squares solutions

Dual analog of minimum length least squares solution

ABSTRACT

This paper deals with properties of dual generalized inverses and uses them in solving linear dual equations. The first part of this paper addresses several properties of dual generalized inverses of dual matrices and provides new theoretical and computational insights. In the second part necessary and sufficient conditions for a dual matrix to have a {1}-dual generalized inverse are obtained. A dual matrix with a rank deficient primal part generically (almost always) is shown to have no {1}-dual generalized inverse and hence, neither a {1,3}-dual generalized inverse or a Moore-Penrose dual generalized inverse. The necessary and sufficient condition to have a solution to a system of linear dual equations is obtained. An explicit expression for the solution, when one exists, is obtained and it is shown not to be unique, in general. The dual analog of the real least-squares problem is obtained through the {1,3}-dual generalized inverse; as with real matrices, it is shown that this least-squares dual solution is not unique, in general. The dual analog of the real minimum-length least-squares solution is obtained through the development of the Moore-Penrose dual generalized inverse. Numerical examples are provided throughout, mainly from kinematics, to illustrate the results obtained.

© 2020 Elsevier Ltd. All rights reserved.

1. Introduction

Dual matrices are used today in a variety of fields like kinematic analysis and synthesis of spatial mechanisms [1–5], and in robotics [6–11]. Their presence is felt in other area of science and engineering as well, such as sensor calibration, and this has sparked an increased recent interest in various aspects of linear algebra and computational approaches related to their usage [12–16]. In many inverse problems in the kinematics and analysis of machines and mechanisms, the Moore-Penrose dual generalized inverse (MPDGI) is employed. Recently, though, it has been shown that unlike all real matrices that always have unique Moore-Penrose inverses, all dual matrices do not have Moore-Penrose dual inverses and therefore caution needs to be exercised in their use in practical applications [17]. In the first part of this paper (Section 2) the properties of dual generalized inverses are explored in greater depth and several new results are obtained. The second part of the paper (Section 3) deals with solutions of linear dual equations using consistent and inconsistent experimental/simulated data—an aspect of considerable importance to the area of kinematics and mechanics. Topics addressed are conditions for the existence of solutions to linear dual equations, dual analogs of least-squares solutions of linear real equations, and dual analogs of the least-squares minimum-norm solutions.

E-mail address: feuUSC@gmail.com

The structure of the paper is as follows. Section 2 proposes a norm of a dual vector and shows its connection to its Euclidean norm. In a step-by-step build-up of the theory of dual matrices, it includes new results on: congruence and similarity transformations of dual matrices, explicit formulae for the determination of different dual generalized inverses for matrices when they exist, properties of the MPDGs of matrices with full-rank primal parts, uniqueness of MPDGs, and closed form expressions for determining MPDGs of dual matrices whose primal parts have full rank.

Section 3 deals with solutions of systems of linear dual equations. New results include: the necessary and sufficient conditions for a dual matrix to have a $\{1\}$ -dual generalized inverse, and hence the necessary conditions for an MPDGI of a dual matrix to exist; genericity of the non-existence of an MPDGI of a dual matrix with rank-deficient primal part; explicit solutions of linear dual matrix equations and their non-uniqueness; the dual analog of least-squares solutions for an inconsistent set of real matrix equations; and, the dual analog of least-squares minimum-norm solutions of real inconsistent equations.

Matrix dual equations arise in kinematic analysis and synthesis and in inverse problems when using experimental/simulated data. Section 4 provides the conclusions and points out some of the more important salient results that are new. Numerical examples are provided along the way to illustrate the results and increase the readability of the paper. All computations are done using the Matlab platform.

2. Dual generalized inverses, their definition and properties

We begin by establishing some notation, which will be used throughout the paper.

2.1. Notation and preliminaries

Consider the m -by- 1 dual vectors $\hat{u}_i = p_i + \varepsilon q_i$, $p_i \neq 0$, $i = 1, 2$, in which ε is the hypercomplex unit basis with $\varepsilon^2 = 0$. For $i = 1, 2$, the m -by- 1 vector p_i is called in this paper the primal (sometimes also called, real) part of \hat{u}_i , and the m -by- 1 vector q_i is called the dual part of \hat{u}_i . We consider their Euclidean norms defined by

$$\|\hat{u}_i\|^2 = \hat{u}_i^T \hat{u}_i = (p_i + \varepsilon q_i)^T (p_i + \varepsilon q_i) = \|p_i\|^2 + 2\varepsilon p_i^T q_i, \quad p_i \neq 0, \quad i = 1, 2, \quad (1)$$

since $\varepsilon^2 = 0$. We have denoted the Euclidean norm of the real vector p_i by $\|p_i\|$. We now define a norm of the dual vector \hat{u}_i , denoted by $\langle \hat{u}_i \rangle$, using the right-most expression in Eq. (1), as

$$\langle \hat{u}_i \rangle = \langle p_i + \varepsilon q_i \rangle = \|p_i\| + \|q_i\|, \quad i = 1, 2. \quad (2)$$

Note the manner in which the vector q_i appears in the structure of the square of the Euclidean norm of the dual vectors \hat{u}_i as seen from the right hand side of Eq. (1). Similarly, to find the norm of the dual vector $\hat{e} = \hat{u}_1 + \hat{u}_2 = p_1 + p_2 + \varepsilon(q_1 + q_2)$ we can first find the square of its Euclidean norm

$$\begin{aligned} \|\hat{e}\|^2 &= \|\hat{u}_1 + \hat{u}_2\|^2 = (\hat{u}_1 + \hat{u}_2)^T (\hat{u}_1 + \hat{u}_2) = \|\hat{u}_1\|^2 + \|\hat{u}_2\|^2 + 2\hat{u}_1^T \hat{u}_2 \\ &= [p_1 + p_2 + \varepsilon(q_1 + q_2)]^T [p_1 + p_2 + \varepsilon(q_1 + q_2)] \\ &= \|p_1 + p_2\|^2 + 2\varepsilon(p_1 + p_2)^T (q_1 + q_2), \end{aligned} \quad (3)$$

and from the last expression above, as before, we extract the norm of the dual vector \hat{e} , like we did using Eq. (1), as

$$\langle \hat{e} \rangle = \langle \hat{u}_1 + \hat{u}_2 \rangle = \|p_1 + p_2\| + \|q_1 + q_2\|. \quad (4)$$

Furthermore,

$$\hat{u}_1^T \hat{u}_2 = (p_1 + \varepsilon q_1)^T (p_2 + \varepsilon q_2) = p_1^T p_2 + \varepsilon(p_1^T q_2 + q_1^T p_2), \quad (5)$$

so that $\hat{u}_1^T \hat{u}_2 = 0$ implies that $p_1^T p_2 = (p_1^T q_2 + q_1^T p_2) = 0$. Thus, when $\hat{u}_1^T \hat{u}_2 = 0$, then $\|p_1 + p_2\|^2 = (p_1 + p_2)^T (p_1 + p_2) = \|p_1\|^2 + \|p_2\|^2$, and relation (3) becomes

$$\begin{aligned} \|\hat{e}\|^2 &= \|\hat{u}_1\|^2 + \|\hat{u}_2\|^2 = \|p_1 + p_2\|^2 + 2\varepsilon(p_1 + p_2)^T (q_1 + q_2) \\ &= \|p_1\|^2 + \|p_2\|^2 + 2\varepsilon(p_1 + p_2)^T (q_1 + q_2) \end{aligned} \quad (6)$$

and the norm of \hat{e} can be extracted from the last expression as

$$\langle \hat{e} \rangle = \sqrt{\|p_1\|^2 + \|p_2\|^2} + \|q_1 + q_2\| \leq \sqrt{\|p_1\|^2 + \|p_2\|^2} + \|q_1\| + \|q_2\| := \langle \hat{e} \rangle_{bound} \quad (7)$$

where $\langle \hat{u}_i \rangle = \|p_i\| + \|q_i\|$, $i = 1, 2$. The quantity denoted by $\langle \hat{e} \rangle_{bound}$ is an upper bound on the norm of the dual vector \hat{e} . Note that when $\hat{u}_2 = 0$, the first equality in Eq. (7) gives $\langle \hat{e} \rangle = \|p_1\| + \|q_1\|$, like it should, since now $\hat{e} = \hat{u}_1$.

The expression for the norm given in Eq. (2) does indeed define a norm because of the following.

- (i) $\langle \hat{u}_i \rangle > 0$, and $\langle \hat{u}_i \rangle = 0$ if and only if $\hat{u}_i = 0$,
- (ii) For a real number α , $\langle \alpha \hat{u}_i \rangle$ is obtained by first taking the square of its Euclidean norm $\|\alpha \hat{u}_i\|^2 = (\alpha p_i + \varepsilon \alpha q_i)^T (\alpha p_i + \varepsilon \alpha q_i) = \|\alpha p_i\|^2 + 2\varepsilon(\alpha p_i)^T (\alpha q_i)$, $p_i \neq 0$, so that the norm of $\alpha \hat{u}_i$ is $\langle \alpha \hat{u}_i \rangle = \|\alpha p_i\| + \|\alpha q_i\| = |\alpha|(\|p_i\| + \|q_i\|) = |\alpha| \langle \hat{u}_i \rangle$, and,

(iii) Using Eq. (4), we get

$$\langle \hat{u}_1 + \hat{u}_2 \rangle = \|p_1 + p_2\| + \|q_1 + q_2\| \leq \|p_1\| + \|q_1\| + \|p_2\| + \|q_2\| \leq \langle \hat{u}_1 \rangle + \langle \hat{u}_2 \rangle.$$

While the norm defined in Eq. (2) can be obviously defined without any reference to the so-called square of the Euclidean norm, the observed connection between the norm and square of the Euclidean norm will be important in what follows.

The dual m -by- n matrix $\hat{A} = A + \varepsilon B$ is the matrix counterpart of the dual vector \hat{u}_1 . As before, the m -by- n matrix A is referred to as the primal part of \hat{A} , and the m -by- n matrix B is referred to as its dual part. In this paper we shall assume that the matrix $A \neq 0$ when $B \neq 0$.

Let us assume that there exists an n -by- m dual matrix \hat{A}^+ such that the following four ordered set of dual relations are satisfied, similar to real matrices,

$$1. \quad \hat{A}\hat{A}^+\hat{A} = \hat{A}, \tag{8}$$

$$2. \quad \hat{A}^+\hat{A}\hat{A}^+ = \hat{A}^+, \tag{9}$$

$$3. \quad \hat{A}\hat{A}^+ = (\hat{A}\hat{A}^+)^T, \text{ and,} \tag{10}$$

$$4. \quad \hat{A}^+\hat{A} = (\hat{A}^+\hat{A})^T. \tag{11}$$

Eqs. (8)–(11) are, respectively, referred as the first, second, third, and fourth Moore-Penrose (MP) dual conditions and any matrix, \hat{A}^+ , that satisfies all these four conditions is called the Moore-Penrose dual generalized inverse (MPDGI, for short) of the matrix \hat{A} .

For a matrix \hat{A} , if instead of \hat{A}^+ in Eq. (8), we find an n -by- m dual matrix \hat{G} that satisfies the first Moore-Penrose condition, namely, $\hat{A}\hat{G}\hat{A} = \hat{A}$, then the matrix \hat{G} is referred to as the {1}-dual generalized inverse of the matrix \hat{A} , and it is denoted by $\hat{A}^{\{1\}}$; since, $\hat{A}^{\{1\}} := \hat{G}$, we then have $\hat{A}\hat{A}^{\{1\}}\hat{A} = \hat{A}$. Likewise, if instead of \hat{A}^+ in Eq. (9), we find some dual matrix \hat{P} that satisfies the second Moore-Penrose condition, namely, $\hat{P}\hat{A}\hat{P} = \hat{P}$, then this dual matrix \hat{P} is referred to as the {2}-dual generalized inverse of the matrix \hat{A} , and it is denoted by $\hat{A}^{\{2\}}$; since $\hat{A}^{\{2\}} := \hat{P}$, the {2}-dual generalized inverse of \hat{A} satisfies the second Moore-Penrose (MP) dual condition $\hat{A}^{\{2\}}\hat{A}\hat{A}^{\{2\}} = \hat{A}^{\{2\}}$.

Similarly, if a matrix \hat{G} satisfies the first and the third MP condition, namely, the two conditions $\hat{A}\hat{G}\hat{A} = \hat{A}$ and $\hat{A}\hat{G} = (\hat{A}\hat{G})^T$ Eqs. (8) and (10), it is called the {1,3}-dual generalized inverse of the matrix \hat{A} , and it is denoted by $\hat{A}^{\{1,3\}}$. The {1,3}-inverse of \hat{A} therefore satisfies the relations $\hat{A}\hat{A}^{\{1,3\}}\hat{A} = \hat{A}$ and $\hat{A}\hat{A}^{\{1,3\}} = (\hat{A}\hat{A}^{\{1,3\}})^T$. And so on.

Since the matrix \hat{A}^+ satisfies all four of the MP conditions, it is referred to as the {1,2,3,4}-dual generalized inverse of the matrix \hat{A} , and is also denoted by $\hat{A}^{\{1,2,3,4\}}$. As stated before, it is often denoted by MPDGI in the literature [3,17].

Remark 1. The 4 MP conditions for real matrices are obtained by removing the ‘hats’ from Eqs. (8)–(11). In what follows the {*-}generalized inverse of a real matrix, where ** stands for one or more distinct elements of set {1,2,3,4}, will be denoted by $A^{(*)}$ [18,19].♦

With this notation we are ready to present the necessary and sufficient conditions that a matrix needs to satisfy for it to have a given type of dual generalized inverse.

2.2. Necessary and Sufficient Conditions for a Dual Matrix to Have Various Types of Dual Generalized Inverses

Consider the m -by- n matrix $\hat{A} = A + \varepsilon B$ and the n -by- m dual matrix $\hat{G} = G + \varepsilon R$. We start by stating a result proved earlier in Refs. [1,17] since it will be used extensively in the sequel and will lay out the notation to be used in the paper.

Result 1. The matrix \hat{G} is a [1,17]:

1 {1}-dual generalized inverse of the matrix \hat{A} , so that $\hat{G} := \hat{A}^{\{1\}}$, if and only if

$$(i) \quad AGA = A, \text{ and } (ii) \quad B = BGA + ARA + AGB \tag{12}$$

2 {2}-dual generalized inverse of the matrix \hat{A} , so that $\hat{G} := \hat{A}^{\{2\}}$, if and only if

$$(i) \quad GAG = G, \text{ and } (ii) \quad R = RAG + GBG + GAR \tag{13}$$

3 {3}-dual inverse of the matrix \hat{A} , so that $\hat{G} := \hat{A}^{\{3\}}$, if and only if

$$(i) \quad AG = (AG)^T, \text{ and } (ii) \quad (BG + AR) = (BG + AR)^T; \text{ and,} \tag{14}$$

4 {4}-dual inverse of the matrix \hat{A} , so that $\hat{G} := \hat{A}^{\{4\}}$, if and only if

$$(i) \quad GA = (GA)^T, \text{ and } (ii) \quad (RA + GB) = (RA + GB)^T. \tag{15}$$

□

Consider the m -by- m square dual matrix $\hat{A} = A + \varepsilon B$, and assume that its dual generalized inverse $\hat{A}^{(*)} = G + \varepsilon R$ exists, where by ** in the superscript we denote one or more distinct elements of the set $\{1,2,3,4\}$. We can think of the matrices A and B as the matrix representations of the linear operators \mathfrak{A} and \mathfrak{B} respectively in the standard orthonormal basis. The matrix representations of these two operators in some other orthonormal basis whose column vectors belong to the matrix T are $T^T A T$ and $T^T B T$. Hence, the dual matrix \hat{A} in this new basis is then given by

$$\tilde{\hat{A}} = T^T \hat{A} T = T^T A T + \varepsilon T^T B T := \tilde{A} + \varepsilon \tilde{B}. \tag{16}$$

We next show that the dual inverse $\tilde{\hat{A}}^{(*)}$ expressed in the new basis is simply given by $T^T G T + \varepsilon T^T R T$.

Result 2. Consider the m -by- m dual matrix $\hat{A} = A + \varepsilon B$ whose primal and dual parts are the matrix representations of the real linear operators \mathfrak{A} and \mathfrak{B} respectively, and whose $\hat{A}^{(*)} = G + \varepsilon R$ exists, where the superscript ** denotes one or more distinct elements of the set $\{1,2,3,4\}$. The matrix representation of this dual matrix \hat{A} in a (any) new real orthonormal basis is $\tilde{\hat{A}} = \tilde{A} + \varepsilon \tilde{B}$ and it is given in Eq. (16). Then the $\{*\}$ -dual generalized inverse of $\tilde{\hat{A}}$ exists and is given by

$$\tilde{\hat{A}}^{(*)} = T^T \hat{A}^{(*)} T = T^T G T + \varepsilon T^T R T := \tilde{G} + \varepsilon \tilde{R} \tag{17}$$

Conversely, if $\tilde{\hat{A}}^{(*)}$ exists for a dual matrix $\tilde{\hat{A}} = \tilde{A} + \varepsilon \tilde{B}$ expressed in a certain orthonormal basis, then $\hat{A}^{(*)}$ exists for the dual matrix $\hat{A} = A + \varepsilon B$ expressed in the standard basis, or in any other orthonormal basis.

Proof: We need to show that if \tilde{A} and \tilde{B} (defined in Eq. (16)) are the primal and dual parts of the dual matrix $\tilde{\hat{A}}$, then its $\{*\}$ -dual generalized inverse (where $\{*\}$ denotes one or more elements of the set $\{1,2,3,4\}$) has \tilde{G} and \tilde{R} as its primal and dual parts.

Let us say that ** contains the number 1 so $\hat{A}^{(1)} = G + \varepsilon R$. For the first MP condition (Eq. (12)) to be satisfied by $\hat{A}^{(1)}$ we need to show that

$$\tilde{A} \tilde{G} \tilde{A} = \tilde{A} \text{ and } \tilde{B} \tilde{G} \tilde{A} + \tilde{A} \tilde{R} \tilde{A} + \tilde{A} \tilde{G} \tilde{B} = \tilde{B}. \tag{18}$$

Noting the definitions given in Eq. (17), the first of the two equations in Eq. (18) is then

$$\tilde{A} \tilde{G} \tilde{A} = T^T A T T^T G T T^T A T = T^T A G A T = T^T A T = \tilde{A}. \tag{19}$$

The third equality above follows because $\hat{A}^{(1)}$ is assumed to have a $\{1\}$ -dual generalized inverse and is given by $G + \varepsilon R$. Similarly the second equation in Eq. (18) becomes

$$\begin{aligned} T^T B T T^T G T T^T A T + T^T A T T^T R T T^T A T + T^T A T T^T G T T^T B T \\ = T^T (B G A + A R A + A G B) T = T^T B T = \tilde{B}. \end{aligned} \tag{20}$$

in which the second equality in the second line follows because \hat{A} is assumed to have a $\{1\}$ -dual generalized inverse.

Were ** to contain the number 3 so that $\hat{A}^{(3)} = G + \varepsilon R$, then the third MP dual condition for $\hat{A}^{(3)}$ requires that

$$\tilde{A} \tilde{G} = (\tilde{A} \tilde{G})^T \text{ and } \tilde{B} \tilde{G} + \tilde{A} \tilde{R} = (\tilde{B} \tilde{G} + \tilde{A} \tilde{R})^T \tag{21}$$

This is true because $\tilde{A} \tilde{G} = T^T A T T^T G T = T^T A G T = T^T (A G)^T T = T^T G^T T T^T A^T T = \tilde{G}^T \tilde{A}^T$. Similarly,

$$\begin{aligned} \tilde{B} \tilde{G} + \tilde{A} \tilde{R} &= T^T B T T^T G T + T^T A T T^T R T = T^T (B G + A R) T \\ &= T^T (B G + A R)^T T = T^T G^T B^T T + T^T R^T A^T T \\ &= T^T G^T T T^T B^T T + T^T R^T T T^T A^T T = \tilde{G}^T \tilde{B}^T + \tilde{R}^T \tilde{A}^T. \end{aligned} \tag{22}$$

One can similarly check the 4th MP dual condition in Eq. (15).

In like manner it is easy to show that, conversely, if $\tilde{\hat{A}}^{(*)}$ exists in a certain orthonormal basis, then $\hat{A}^{(*)}$ exists in the standard basis, and therefore in any other orthonormal basis. Hence the result. \square

Remark 2. If the new real basis in Eq. (16) is not orthonormal, then the matrix representation of the dual matrix \hat{A} in the new basis becomes

$$\tilde{\hat{A}} = T^{-1} \hat{A} T = T^{-1} A T + \varepsilon T^{-1} B T := \tilde{A} + \varepsilon \tilde{B}.$$

If \hat{A} has a dual inverse $\hat{A}^{(*)} = G + \varepsilon R$ where ** belongs to one or more distinct members of the set $\{1,2\}$, then $\tilde{\hat{A}}$ has the dual inverse $\tilde{\hat{A}}^{(*)} = T^{-1} G T + \varepsilon T^{-1} R T := \tilde{G} + \varepsilon \tilde{R}$. This can be easily seen by replacing T^T by T^{-1} in Eqs. (19) and (20). \blacklozenge

We have shown that if a square dual matrix that has a dual generalized inverse (of any kind) is represented in an (any) orthonormal basis different from the standard basis, its dual generalized inverse in the new basis is the matrix representation of its old generalized inverse expressed in the new basis. We have also shown that the square dual matrix $\hat{A} = A + \varepsilon B$ has the dual generalized inverse $A^{(*)}$ if and only if it has a $\{*\}$ -dual generalized inverse in any (every) other orthonormal basis. Furthermore, if we restrict attention to only the $\{1\}$ -, $\{2\}$ -, and $\{1, 2\}$ -dual generalized inverses of square matrices, we

do not need to restrict the new basis to be orthonormal. We use these ideas in the example that follows and in examples later on that appear in kinematic analysis.

Example 1. Consider the matrix

$$\hat{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} := A + \varepsilon B, \tag{23}$$

which when expressed in the non-orthonormal basis whose columns are given in (shown accurate to only 4 places beyond the decimal point)

$$T = \begin{bmatrix} 0.4082 & 0.7071 & -0.3015 \\ 0.4082 & -0.7071 & -0.3015 \\ 0.8165 & -0.0000 & 0.9045 \end{bmatrix} \tag{24}$$

becomes

$$\tilde{\hat{A}} = T^{-1}AT + \varepsilon T^{-1}BT = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 18 & -4.1569 & 9.9704 \\ -1.1547 & 0.0000 & -0.2132 \\ -4.0620 & 1.4071 & -3.0000 \end{bmatrix}. \tag{25}$$

Since the dual matrix $\tilde{\hat{A}}$ in Eq. (25) does not generically have a {1}-dual generalized inverse, and we shall see later on why that is so (see Example 5), we conclude that the matrix \hat{A} also does not generically have a {1}-dual generalized inverse, and therefore no Moore-Penrose dual generalized inverse. Note that the primal part of the dual matrix in Eq. (25) takes a much simpler form in the transformed basis as shown in Eq. (25), and that its rank is now more clearly seen to be 2. \blacklozenge

Result 3. Given an m -by- n dual matrix $\hat{A} = A + \varepsilon B$ the necessary and sufficient conditions that the n -by- m {1,3}-dual generalized inverse, $\hat{A}^{(1,3)} = G + \varepsilon R$, must satisfy are the four conditions in Eqs. (12) and (14). This result will be used in Section 3.

Proof: For the matrix $\hat{A}^{(1,3)}$ to be a {1,3}-dual generalized inverse, it must satisfy the first and the fourth MP condition, namely,

$$\hat{A}\hat{A}^{(1,3)}\hat{A} = \hat{A}, \text{ and}, \tag{26}$$

$$\hat{A}\hat{A}^{(1,3)} = (\hat{A}\hat{A}^{(1,3)})^T. \tag{27}$$

Hence, from Result 1, Parts 1 and 3, the necessary and sufficient conditions for this are

$$AGA = A, AG = (AG)^T \text{ and} \tag{28}$$

$$B = BGA + ARA + AGB, \text{ and } BG + AR = (BG + AR)^T, \tag{29}$$

given in Eqs. (12) and (14).

Remark 3. The first two conditions in Eq. (28) placed on the real matrix G , define a {1,3}-inverse of the real matrix A , denoted by $G = A^{(1,3)}$ (see Remark 1). Hence,

$$\hat{A}^{(1,3)} = A^{(1,3)} + \varepsilon R \tag{30}$$

where the matrix R satisfies the two relations

$$B = BA^{(1,3)}A + ARA + AA^{(1,3)}B, \text{ and } BA^{(1,3)} + AR = (BA^{(1,3)} + AR)^T \tag{31}$$

This result was first developed in Ref. [17]; it is used extensively in the following section of the paper and is therefore provided for increasing the readability of the material that follows. \blacklozenge

Result 4. Given an m -by- n dual matrix $\hat{A} = A + \varepsilon B$ the necessary and sufficient conditions that the n -by- m Moore-Penrose dual generalized inverse, $\hat{A}^+ := \hat{A}^{(1,2,3,4)} = G + \varepsilon R$, must satisfy are the eight conditions in Eqs. (12)–(15).

Proof: For the matrix \hat{A}^+ to be the Moore-Penrose dual generalized inverse (MPDGI) of \hat{A} , it must satisfy all the four MP dual conditions. The necessary and sufficient conditions for this are given in Eqs. (12)–(15). As seen from the first relation in each of the Eqs. (12)–(15), the matrix G satisfies all the four MP conditions for a real matrix (see Remark 1), and is therefore the {1,2,3,4}-inverse of the real matrix A . \square

Remark 4. It should be noted that the requirements given in Eqs. (12)–(15) that are placed on the dual part of \hat{A}^+ may be overly stringent, so that for certain dual matrices \hat{A} , there may exist no matrix \hat{A}^+ that can satisfy these requirements. In that case the matrix \hat{A} has no MPDGI, and the MPDGI of \hat{A} is said not to exist. Example 1 shows a dual matrix whose {1}-dual generalized inverse does not exist, and therefore whose MPDGI does not exist. Ref. [17] shows through a constructive proof

that there are numerous matrices for which no MPDGI exists. Later, in Section 3.1 we will provide a necessary condition for the MPDGI of a dual matrix to exist. ♦

Example 2. We provide an example from kinematics in which three line-vectors from points \mathbf{p}_i to \mathbf{q}_i $i = 1, 2, 3$, are drawn on a flat plate that lies in the plane $y = 2$ in an inertial coordinate frame. The points have coordinates

$$\mathbf{p}_1 = (1, 2, 2), \mathbf{p}_2 = (3, 2, 2), \mathbf{p}_3 = (2, 2, 3) \text{ and}$$

$$\mathbf{q}_1 = (3, 2, 3), \mathbf{q}_2 = (4, 2, 3), \mathbf{q}_3 = (5, 2, 5).$$

The dual matrix of line-vectors is then

$$\hat{A} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \varepsilon \begin{bmatrix} 2 & 2 & 4 \\ 3 & -1 & 5 \\ -4 & -2 & -6 \end{bmatrix}.$$

This matrix has no MPDGI. In fact, no matter what the dual part of A might be, the MPDGI of \hat{A} almost always does not exist [17].

Another simple example of a dual matrix with no MPDGI is

$$\hat{A} = A + \varepsilon B = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & 2 & 3 & 4 \\ 6 & 10 & 5 & 6 \\ 2 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

If the $B(2,2)$ element of the dual part of \hat{A} is different from zero, then the MPDGI of \hat{A} does *not* exist. For a proof of this statement, see Ref. [17]. Any dual matrix with this primal part too does not generically (almost always) have an MPDGI. ‘ #

Remark 5. Consider again the dual matrix $\hat{A} = A + \varepsilon B$. As seen from the first relation in each of the Eqs. (12)–(15), the real part, G , of the matrix $\hat{A}^+ := \hat{A}^{\{1,2,3,4\}}$, assuming it exists, has to satisfy the relations

$$AGA = A, GAG = G, AG = (AG)^T, \text{ and } GA = (GA)^T. \tag{32}$$

All the matrices in Eq. (32) are real and we see from Eq. (32) that matrix G is then the Moore-Penrose inverse of the real matrix A which we denote by A^+ or $A^{\{1,2,3,4\}}$. Hence,

$$\hat{A}^{\{1,2,3,4\}} := \hat{A}^+ = G + \varepsilon R = A^+ + \varepsilon R. \tag{33}$$

For the MPDGI of n -by- m matrix \hat{A}^+ to exist, the matrix R must then satisfy the second relation given in each of the four Eqs. (12)–(15), namely,

$$BA^+A + ARA + AA^+B - B = 0 \tag{34}$$

$$RAA^+ + A^+BA^+ + A^+AR - R = 0 \tag{35}$$

$$BA^+ + AR = (BA^+ + AR)^T, \text{ and,} \tag{36}$$

$$RA + A^+B = (RA + A^+B)^T, \tag{37}$$

since $G = A^+$.

Eq. (33) says that the primal part of the MPDGI, $\hat{A}^{\{1,2,3,4\}}$, of the dual matrix \hat{A} is the Moore-Penrose inverse of the primal part of \hat{A} .

Since $G = A^+$, the primal part of the MPDGI of any dual matrix *always* exists. This is because for any m -by- n real matrix A , its Moore-Penrose inverse exists, and is unique. Thus, when a matrix does not have an MPDGI it simply means that there exists no n -by- m matrix R such that Eqs. (34)–(37) are satisfied. ♦

Remark 6. From Eq. (32) we saw that the $\{1,2,3,4\}$ -dual generalized inverse of the m -by- n matrix $\hat{A} = A + \varepsilon B$, if such an inverse exists, must have its primal part, $G = A^{\{1,2,3,4\}} = A^+$. That is, the primal part of $\hat{A}^{\{1,2,3,4\}}$ is simply $A^{\{1,2,3,4\}}$. From Part (i) of Eqs. (12)–(15), we see that the same is true of any dual generalized inverse, if it exists. For example, the primal part of $\hat{A}^{\{1\}}$ is simply $A^{\{1\}}$; the primal part of $\hat{A}^{\{1,3\}}$ is $A^{\{1,3\}}$; the primal part of $\hat{A}^{\{1,4\}}$ is $A^{\{1,4\}}$; the primal part of $\hat{A}^{\{1,2,3\}}$ is $A^{\{1,2,3\}}$, and so on. In the notation used in Result 2, the primal part of $\hat{A}^{\{**\}}$ is $A^{\{**\}}$, where $**$ stands for one or more distinct elements of the set $\{1,2,3,4\}$. ♦

Result 5. Let the dual m -by- n matrix $\hat{A} = A + \varepsilon B$ be given. If the MPDGI, $\hat{A}^+ = A^+ + \varepsilon R$, of \hat{A} exists, then \hat{A}^+ is unique.

Proof: Assume that the MPDGI of \hat{A} exists and is $\hat{A}^+ = A^+ + \varepsilon R$. Suppose that another distinct MPDGI of \hat{A} exists that is distinct from this MPDGI. Then by Remark 5, this other MPDGI must have the form $A^+ + \varepsilon \tilde{R}$ with $X := R - \tilde{R} \neq 0$ because the primal part of every MPDGI of $\hat{A} = A + \varepsilon B$ must be A^+ , which is unique since A is real. So, all we need to prove is that if a real n -by- m matrix R that satisfies Eqs. (34)–(37) exists, then R is unique.

From Eq. (34), the real matrices R and \tilde{R} must satisfy the relations

$$B = BA^+A + ARA + AA^+B, \text{ and } B = BA^+A + A\tilde{R}A + AA^+B. \tag{38}$$

Since the matrices A , B , and A^+ are known, upon subtracting the second equation from the first we get

$$0 = A(R - \tilde{R})A = AXA. \tag{39}$$

From Eq. (35), we see that R and \tilde{R} must satisfy the relations

$$R = RAA^+ + A^+BA^+ + A^+AR, \text{ and } \tilde{R} = \tilde{R}AA^+ + A^+BA^+ + A^+A\tilde{R}. \tag{40}$$

Subtracting the second equation from the first then gives

$$X = (R - \tilde{R}) = XAA^+ + A^+AX. \tag{41}$$

Similarly, from Eq. (36) R and \tilde{R} must satisfy the equations

$$(BA^+ + AR) = (BA^+ + AR)^T, \text{ and } (BA^+ + A\tilde{R}) = (BA^+ + A\tilde{R})^T. \tag{42}$$

Upon subtracting the second equation from the first we get

$$AX = (AX)^T. \tag{43}$$

Using Eq. (37) gives, likewise, the relation

$$XA = (XA)^T. \tag{44}$$

Thus, the matrix X that we have assumed to be nonzero, because the two MPDGIs are assumed to be distinct, must satisfy Eqs. (39), (41), (43), and (44).

Post-multiplying both sides of Eq. (39) by A^+ we get

$$\begin{aligned} 0 &= AXAA^+ = (AX)^TAA^+ = X^T A^T AA^+ \\ &= X^T A^T (AA^+)^T = X^T A^T (A^+)^T A^T \\ &= X^T (AA^+A)^T = X^T A^T. \end{aligned} \tag{45}$$

In the second equality in the first line above we have used Eq. (43); in the first equality in the second line we have used the third MP condition for a real matrix; and in the last equality in the third line we have used the first MP condition for a real matrix (see Remark 1). Taking the transpose of Eq. (45) we find that the matrix X must satisfy the relation

$$AX = 0. \tag{46}$$

Similarly, pre-multiplying Eq. (39) by A^+ we get

$$\begin{aligned} 0 &= A^+AXA = A^+A(XA)^T = A^+AA^T X^T \\ &= (A^+A)^T A^T X^T = A^T (A^+)^T A^T X^T \\ &= (AA^+A)^T X^T = A^T X^T. \end{aligned} \tag{47}$$

Again taking the transpose of Eq. (47) yields an additional relation that X must satisfy, which is

$$XA = 0. \tag{48}$$

Eqs. (46) and (48) when used in Eq. (41) yield

$$X = XAA^+ + A^+AX = 0. \tag{49}$$

which contradicts our assumption that $X \neq 0$. Hence, if a matrix R satisfies Eqs. (34)–(37) it must be unique, and the MPDGI, $\hat{A}^+ = A^+ + \varepsilon R$, of a dual matrix \hat{A} , if it exists, must be unique. \square

Example 3. Once again consider a rigid body on which line-vectors are drawn from points

$$\mathbf{p}_1 = (1, 0, 2), \mathbf{p}_2 = (0, 1, 1), \mathbf{p}_3 = (1, 2, 0), \mathbf{p}_4 = (1, 2, 1),$$

respectively, to the points

$$\mathbf{q}_1 = (3, 1, 1), \mathbf{q}_2 = (2, 3, 2), \mathbf{q}_3 = (3, 2, 1), \mathbf{q}_4 = (1, 0, 1).$$

The matrix $\hat{A} = A + \varepsilon B$ that describes these line-vectors from \mathbf{p}_i to $\mathbf{q}_i, i = 1, \dots, 4$, is given by

$$A = \begin{bmatrix} 2 & 2 & 2 & 0 \\ 1 & 2 & 0 & -2 \\ -1 & 1 & 1 & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & 1 & -2 & -2 \\ -5 & -2 & 1 & 0 \\ 1 & 2 & 4 & 2 \end{bmatrix}. \tag{50}$$

Its Moore-Penrose inverse (MPDGI) is given by $\hat{A}^+ = A^+ + \varepsilon R$ in which

$$A^+ = \begin{bmatrix} 2.4999999999999999e - 01 & 1.526556658859590e - 16 & -4.9999999999999999e - 01 \\ 4.166666666666667e - 02 & 1.666666666666666e - 01 & 2.5000000000000001e - 01 \\ 2.0833333333333334e - 01 & -1.666666666666667e - 01 & 2.4999999999999998e - 01 \\ 1.666666666666667e - 01 & -3.333333333333333e - 01 & -2.775557561562891e - 17 \end{bmatrix},$$

and

$$R = \begin{bmatrix} 8.020833333333336e - 01 & -7.916666666666664e - 01 & 8.124999999999990e - 01 \\ -5.034722222222235e - 02 & 9.027777777777779e - 02 & -5.937499999999998e - 01 \\ -6.475694444444441e - 01 & 1.180555555555549e - 01 & 4.062500000000005e - 01 \\ -2.118055555555553e - 01 & -5.55555555555558e - 01 & 9.374999999999997e - 01 \end{bmatrix}$$

The 2-norm of the errors in satisfying the conditions for the MP inverse by A^+ is $O(8 \times 10^{-16})$ and the 2-norm of the errors in satisfying Eqs. (34)–(37) by R is of the order of $O(2 \times 10^{-15})$. \aleph

Remark 7. Writing out the matrix entries in the above two matrices using Matlab’s ‘long e’ format clearly uses up considerable space. Displaying each matrix entry to this level of precision in subsequent computational results that are shown for purposes of illustration would make matrix equations with such matrices both unwieldy and difficult to grasp. Hence, for all computations below, only four digits beyond the decimal point in the matrix entries are displayed.

It should be emphasized that caution is required in directly using the numbers given in these matrix elements (truncated to only 4 decimal point accuracy) in computations, since their limited precision can cause round off errors making it appear that the 4 dual conditions are not satisfied, as happens in Example 3 above. The computed results for each illustrative example shown below—whose truncated versions up to only 4 decimal point accuracy are exhibited—are accurate, and they satisfy the dual conditions, where required, with errors of $O(10^{-14})$, or less. Wherever such computations are displayed below, the reader is reminded of this cautionary remark. \blacklozenge

We next present a central result on Moore-Penrose dual generalized inverses that will be used later. It is known that if \hat{A}^+ exists, then $(\hat{A}^T)^+$ exists (see, Ref. [17]).

Result 6. If the Moore-Penrose dual generalized inverse, \hat{A}^+ , of the m -by- n dual matrix $\hat{A} = A + \varepsilon B$ exists, then

$$i. \quad (\hat{A}^T)^+ = (\hat{A}^+)^T \tag{51}$$

$$ii. \quad (\hat{A}\hat{A}^T)^+ = (\hat{A}^+)^T \hat{A}^+ = (\hat{A}^T)^+ \hat{A}^+ \tag{52}$$

$$iii. \quad \hat{A}^+ = (\hat{A}^T \hat{A})^+ \hat{A}^T = \hat{A}^T (\hat{A} \hat{A}^T)^+ \tag{53}$$

Proof:

(i) Denoting $\hat{C} := \hat{A}^T$, the matrix $\hat{C}^+ := (\hat{A}^T)^+$ satisfies the four MP dual conditions given in Eqs. (8)–(11). Hence

$$\hat{A}^T \hat{C}^+ \hat{A}^T = \hat{A}^T, \hat{C}^+ \hat{A}^T \hat{C}^+ = \hat{C}^+; \hat{A}^T \hat{C}^+ \text{ and } \hat{C}^+ \hat{A}^T \text{ are symmetric.} \tag{54}$$

To prove (51) we need to show that these relations are also satisfied when \hat{C}^+ in Eq. (54) is replaced by $(\hat{A}^+)^T$, which is the right-hand side of Eq. (51). Doing this, we get

$$\hat{A}^T (\hat{A}^+)^T \hat{A}^T = (\hat{A} \hat{A}^+ \hat{A})^T = \hat{A}^T, (\hat{A}^+)^T \hat{A}^T (\hat{A}^+)^T = (\hat{A}^+ \hat{A} \hat{A}^+)^T = (\hat{A}^+)^T, \text{ and} \tag{55}$$

$$\hat{A}^T (\hat{A}^+)^T = (\hat{A}^+ \hat{A})^T = \hat{A}^+ \hat{A} = \left[\hat{A}^T (\hat{A}^+)^T \right]^T, (\hat{A}^+)^T \hat{A}^T = (\hat{A} \hat{A}^+)^T = \left[(\hat{A}^+)^T \hat{A}^T \right]^T \tag{56}$$

which proves the result. The last equality in Eq. (52) follows from Eq. (51).

(ii) Denoting now $\hat{C} := \hat{A} \hat{A}^T$ the four MP dual conditions satisfied by \hat{C}^+ (assuming it exists) are

$$\hat{A} \hat{A}^T \hat{C}^+ \hat{A} \hat{A}^T = \hat{A} \hat{A}^T, \hat{C}^+ \hat{A} \hat{A}^T \hat{C}^+ = \hat{C}^+; \hat{A} \hat{A}^T \hat{C}^+ \text{ and } \hat{C}^+ \hat{A} \hat{A}^T \text{ are symmetric.} \tag{57}$$

We need to show that when \hat{C}^+ is replaced in Eq. (57) by $(\hat{A}^+)^T \hat{A}^+$, these conditions are satisfied. Doing this, we get

$$\underbrace{\hat{A} \hat{A}^T (\hat{A}^+)^T}_{=\hat{A}^+ \hat{A}} \hat{A}^+ \hat{A} \hat{A}^T = \hat{A} \underbrace{\hat{A}^+ \hat{A} \hat{A}^+}_{\hat{A}^+} \hat{A} \hat{A}^T = \hat{A} \hat{A}^T, \tag{58}$$

$$(\hat{A}^+)^T \underbrace{\hat{A}^+ \hat{A}}_{=(\hat{A}^+ \hat{A})^T = \hat{A}^T (\hat{A}^+)^T} \hat{A}^T (\hat{A}^+)^T \hat{A}^+ = (\hat{A}^+)^T \underbrace{\hat{A}^T (\hat{A}^+)^T \hat{A}^T}_{A^T} (\hat{A}^+)^T \hat{A}^+ = (A^+)^T \hat{A}^+, \tag{59}$$

$$\hat{A} \hat{A}^T (\hat{A}^+)^T \hat{A}^+ = \hat{A} \hat{A}^+ \hat{A} \hat{A}^+ = \hat{A} \hat{A}^+, \text{ which is symmetric, and} \tag{60}$$

$$(\hat{A}^+)^T \underbrace{\hat{A}^+ \hat{A}}_{=\hat{A}^T (A^+)^T} \hat{A}^T = (\hat{A}^+)^T \hat{A}^T (\hat{A}^+)^T \hat{A}^T = \hat{A} \hat{A}^+, \text{ which is symmetric.} \tag{61}$$

Hence, the result.

(iii) By replacing \hat{A}^+ in Eqs. (8)–(11) with $(\hat{A}^T \hat{A})^+ \hat{A}^T$ (or with $\hat{A}^T (\hat{A} \hat{A}^T)^+$), we can show, that these four equations are satisfied. We illustrate the proof for the first equality in (iii) by replacing \hat{A}^+ by $(\hat{A}^T \hat{A})^+ \hat{A}^T$ in Eq. (8), which gives

$$\hat{A} (\hat{A}^T \hat{A})^+ \hat{A}^T \hat{A} = \hat{A} \hat{A}^+ \underbrace{(\hat{A}^T)^+ \hat{A}^T}_{=(AA^+)^T} \hat{A} = \hat{A} \hat{A}^+ \underbrace{(\hat{A} \hat{A}^T)^+}_{=\hat{A} \hat{A}^+} \hat{A} = \hat{A} \hat{A}^+ \hat{A} \hat{A}^+ \hat{A} = \hat{A}. \tag{62}$$

In the first equality above we have used Eq. (52).

Replacing \hat{A}^+ by $(\hat{A}^T \hat{A})^+ \hat{A}^T$ in Eq. (9) we get

$$\underbrace{(\hat{A}^T \hat{A})^+}_{\hat{A}^+ (\hat{A}^T)^+} \hat{A}^T \hat{A} \underbrace{(\hat{A}^T \hat{A})^+}_{\hat{A}^+ (\hat{A}^T)^+} \hat{A}^T = \hat{A}^+ \underbrace{(\hat{A}^T)^+}_{(\hat{A} \hat{A}^T)^+} \hat{A}^T \hat{A} \hat{A}^+ \underbrace{(\hat{A}^T)^+}_{(\hat{A} \hat{A}^T)^+} \hat{A}^T = \hat{A}^+ \hat{A} \hat{A}^+ \hat{A} \hat{A}^+ \hat{A} \hat{A}^+ = \hat{A}^+ \hat{A} \hat{A}^+ = \hat{A}^+. \tag{63}$$

Similarly, Eq. (10) yields

$$\hat{A} (\hat{A}^T \hat{A})^+ \hat{A}^T = \hat{A} \hat{A}^+ \underbrace{(\hat{A}^+)^T}_{(\hat{A} \hat{A}^T)^+} \hat{A}^T = \hat{A} \hat{A}^+ \hat{A} \hat{A}^+ = \hat{A} \hat{A}^+, \tag{64}$$

which is symmetric, and Eq. (11) yields

$$(\hat{A}^T \hat{A})^+ \hat{A}^T \hat{A} = \hat{A}^+ \underbrace{(\hat{A}^+)^T}_{(\hat{A} \hat{A}^T)^+} \hat{A}^T \hat{A} = \hat{A}^+ \hat{A} \hat{A}^+ \hat{A} = \hat{A}^+ \hat{A}, \tag{65}$$

which is also symmetric. Hence the result.

The second equality in (iii) can be proved in like manner. \square

Since in many applications of dual inverses, such as found in the area of kinematics, the real part, A , of the dual matrix \hat{A} often has full row or full column rank, we give below results pertaining to that situation.

Result 7. (i) If the primal part, A , of the dual matrix $\hat{A} = A + \varepsilon B$ is an m -by- n matrix has rank m , then a $\{1,2,3\}$ -dual generalized matrix of \hat{A} always exists, and is given by

$$\hat{A}^{\{1,2,3\}} = A^+ - \varepsilon A^+ B A^+ := G + \varepsilon R \tag{66}$$

(ii) If the primal part, A , of the dual matrix $\hat{A} = A + \varepsilon B$ is an m -by- n matrix with rank n , then a $\{1,2,4\}$ -dual generalized matrix of \hat{A} always exists, and is given by

$$\hat{A}^{\{1,2,4\}} = A^+ - \varepsilon A^+ B A^+ := G + \varepsilon R \tag{67}$$

Proof: (i) According to Remark 6, the primal part of the dual matrix $\hat{A}^{\{1,2,3\}}$ is $A^{\{1,2,3\}}$. Since every A^+ belongs to the set of matrices $A^{\{1,2,3\}}$, all we need is to show that Eqs. (34)–(36) are satisfied.

We begin by noting that the singular value decomposition of A is given by

$$A = U \Lambda V^T \text{ and } A^+ = V \Lambda^{-1} U^T \tag{68}$$

and because A is a matrix of rank m , the matrices U and Λ are m -by- m ; U is orthogonal, and Λ is a diagonal positive definite m by m matrix. Hence,

$$I - AA^+ = I - U \Lambda V^T V \Lambda^{-1} U^T = I - UU^T = 0. \tag{69}$$

Assume that $\hat{A}^{\{1,2,3\}}$ is as shown in Eq. (66) so that the matrix $R = -A^+ B A^+$.

Eq. (34) is then satisfied since on substituting for R and using Eq. (69) we get

$$\begin{aligned} BA^+ A + ARA + AA^+ B - B &= BA^+ A - AA^+ B A^+ A + AA^+ B - B \\ &= AA^+ B (I - A^+ A) - B (I - A^+ A) \\ &= (I - AA^+) B A^+ A - (I - AA^+) B, \end{aligned} \tag{70}$$

which is zero because of Eq. (69). Also, the left-hand side of Eq. (35) gives

$$\begin{aligned} RAA^+ + A^+ B A^+ + A^+ A R - R &= -A^+ B A^+ A A^+ + A^+ B A^+ - A^+ A A^+ B A^+ + A^+ B A^+ \\ &= A^+ B A^+ (I - AA^+) + A^+ (I - AA^+) B A^+ \\ &= A^+ B (I - A^+ A) A^+ + (I - A^+ A) A^+ B A^+, \end{aligned} \tag{71}$$

which by Eq. (69) also equals zero, as is required. The left-hand side of Eqs. (36) becomes

$$BA^+ + AR = BA^+ - AA^+ B A^+ = (I - AA^+) B A^+ = 0. \tag{72}$$

Hence, Eqs. (34)–(36) are satisfied. The result therefore follows. Note that the fourth Moore-Penrose condition is not satisfied, in general, since $I - A^+ A$ is not zero as required by it (see below). Hence it is not a Moore-Penrose inverse of \hat{A} .

(ii) When A has full column rank n , in the singular value decomposition of the matrix A given in Eq. (68) the matrices V and Λ are n -by- n matrices, and V is an orthogonal matrix.

Thus,

$$I - A^+A = I - V \underbrace{\Lambda^{-1} U^T U \Lambda}_{I} V^T = I - VV^T = 0. \tag{73}$$

As before, all we need is to show that Eqs. (34), (35), and (37) are now satisfied. Using Eq. (73) in Eqs. (70) and (71), we see that Eqs. (34) and (35) are satisfied. Eq. (37) is

$$RA + A^+B = -A^+BA^+A + A^+B = A^+B(I - A^+A) = 0. \tag{74}$$

Hence the result. Note that $(I - AA^+) \neq 0$, in general, and hence it is not a {3}-dual inverse of \hat{A} and thus not a Moore-Penrose inverse of \hat{A} . \square

Recall that a matrix that is a {1,2,3}-dual generalized inverse is also a {1,2}-dual generalized inverse, and a matrix that is a {1,2}-dual generalized inverse is also a {1}-dual generalized inverse. Thus, from Result 7 we see that given any dual m -by- n matrix $\hat{A} = A + \varepsilon B$ that has full row rank or full column rank, its dual generalized inverse $\hat{A}^{[*]}$ always exists, where $^{[*]}$ consists of one or more distinct members of the set {1,2}; and such a dual generalized inverse is given by $\hat{A}^{[*]} = A^+ - \varepsilon A^+BA^+$.

Note that when the row rank of the primal part of the m -by- n dual matrix \hat{A} is less than m , then the last equality in Eq. (69) is no longer valid, and therefore Eq. (72) cannot be satisfied. We cannot then guarantee that a {1,2,3}-dual generalized inverse of \hat{A} always exists. Similarly, when the column rank of the primal part of \hat{A} is less than n , then the last equality in Eq. (73) is no longer valid, and therefore Eq. (74) cannot be satisfied. Once again, we cannot then guarantee that a {1,2,4}-dual generalized inverse of \hat{A} always exists.

We next particularize our results for the case when $m = n$ so that the primal part of the dual matrix \hat{A} is nonsingular, a situation that often arises in kinematics of mechanisms and robotic systems making this result applicable and useful especially to these fields.

Corollary 1. When the primal part, A , of the dual matrix $\hat{A} = A + \varepsilon B$ is a square nonsingular matrix, then the (unique) Moore-Penrose inverse of \hat{A} always exists and is

$$(i) \quad \hat{A}^{-1} = \hat{A}^+ = A^{-1} - \varepsilon A^{-1}BA^{-1}. \tag{75}$$

This defines the inverse of a dual matrix that has a nonsingular primal part.

For two dual m -by- m matrices $\hat{A}_i = A_i + \varepsilon B_i$, $i = 1, 2$, whose primal parts are square and nonsingular

$$(ii) \quad (\hat{A}_1\hat{A}_2)^{-1} = \hat{A}_2^{-1}\hat{A}_1^{-1}. \tag{76}$$

Proof:

(i) The first result is well-know and it can also be easily deduced; it is stated here because it is used later in obtaining Result 8.

(ii) To prove Eq. (76) we see that $\hat{A}_i^{-1} = A_i^{-1} - \varepsilon A_i^{-1}B_iA_i^{-1}$, $i = 1, 2$. The product

$$\hat{A}_1\hat{A}_2 = A_1A_2 + \varepsilon(B_1A_2 + A_1B_2)$$

and since A_1 and A_2 are each nonsingular matrices, their product A_1A_2 is also square and nonsingular. Hence $\hat{A}_1\hat{A}_2$ has an inverse, and by the result in Eq. (75)

$$\begin{aligned} (\hat{A}_1\hat{A}_2)^{-1} &= (A_1A_2)^{-1} - \varepsilon(A_1A_2)^{-1}(B_1A_2 + A_1B_2)(A_1A_2)^{-1} \\ &= A_2^{-1}A_1^{-1} - \varepsilon(A_2^{-1}A_1^{-1}B_1A_1^{-1} + A_2^{-1}B_2A_2^{-1}A_1^{-1}), \end{aligned} \tag{77}$$

which is what is also obtained by taking the product $\hat{A}_2^{-1}\hat{A}_1^{-1}$. \square

In kinematic analysis we often get dual matrices whose primal parts have either full row rank or full column rank, and it is often useful to obtain in closed form the MPDGI of such dual matrix matrices. The next result shows this by utilizing Result 6 and Corollary 1. The Moore-Penrose dual generalized inverse is obtained, without recourse to singular value decompositions and/or complex computational procedures, solely in terms of the primal and dual parts A and B , respectively, of the dual matrix \hat{A} .

Result 8. When the primal part, A , of the m -by- n matrix $\hat{A} = A + \varepsilon B$ has full row rank or full column rank, its unique Moore-Penrose dual generalized inverse, \hat{A}^+ , if it exists, is explicitly given by the following relations.

(i) When $Rank(A) = m$, then

$$\hat{A}^+ = A^TC + \varepsilon[B^TC - A^TDC], \text{ where } C = (AA^T)^{-1} \text{ and } D = BA^T + AB^T; I - \hat{A}\hat{A}^+ = 0. \tag{78}$$

(ii) When $Rank(A) = n$, then

$$\hat{A}^+ = CA^T + \varepsilon[CB^T - CDCA^T], \text{ where } C = (A^TA)^{-1} \text{ and } D = B^TA + A^TB; I - \hat{A}^+\hat{A} = 0. \tag{79}$$

Proof: (i) Let $\text{Rank}(A) = m$. Then the dual matrix

$$\hat{A}\hat{A}^T = (A + \varepsilon B)(A^T + \varepsilon B^T) = AA^T + \varepsilon \underbrace{(BA^T + AB^T)}_D := AA^T + \varepsilon D, \tag{80}$$

so that the primal part of $\hat{A}\hat{A}^T$ is an m -by- m matrix of rank m . By [Corollary 1](#), it therefore has a Moore-Penrose inverse given by

$$(\hat{A}\hat{A}^T)^+ = (\hat{A}\hat{A}^T)^{-1} = (AA^T)^{-1} - \varepsilon(AA^T)^{-1}D(AA^T)^{-1} \tag{81}$$

and by using [Eq. \(53\)](#) from [Result 6](#), we get

$$\hat{A}^+ = \hat{A}^T(\hat{A}\hat{A}^T)^+ = \hat{A}^T \left[\underbrace{(AA^T)^{-1}}_C - \varepsilon(AA^T)^{-1}D(AA^T)^{-1} \right] = \hat{A}^T[C - \varepsilon CDC]. \tag{82}$$

Note that we have obtained the MPDGI of the dual matrix \hat{A} as a product of two dual matrices, which gives the explicit relation

$$\hat{A}^+ = A^T C + \varepsilon [B^T C - A^T CDC], \tag{83}$$

where the m -by- m matrices $C = (AA^T)^{-1}$ and $D = BA^T + AB^T$.

Also,

$$\hat{A}\hat{A}^+ = \hat{A}\hat{A}^T(\hat{A}\hat{A}^T)^+ = \hat{A}\hat{A}^T(\hat{A}\hat{A}^T)^{-1} = I, \tag{84}$$

in which the first equality follows from [Eq. \(53\)](#), and the second follows from the first equality in [Eq. \(81\)](#).

(ii) In a similar way, when the rank of A is n , and the primal part, $A^T A$, of the dual matrix $\hat{A}^T \hat{A}$ is an n -by- n matrix of rank n . We then have

$$\hat{A}^T \hat{A} = A^T A + \varepsilon \underbrace{(B^T A + A^T B)}_D := A^T A + \varepsilon D, \tag{85}$$

and from [Corollary 1](#) we get

$$(\hat{A}^T \hat{A})^+ = (\hat{A}^T \hat{A})^{-1} = (A^T A)^{-1} - \varepsilon(A^T A)^{-1}D(A^T A)^{-1}. \tag{86}$$

Using [Eq. \(53\)](#) we get

$$\hat{A}^+ = (\hat{A}^T \hat{A})^+ \hat{A}^T = \left[\underbrace{(A^T A)^{-1}}_C - \varepsilon(A^T A)^{-1}D(A^T A)^{-1} \right] \hat{A}^T = [C - \varepsilon CDC] \hat{A}^T.$$

which simplifies to the explicit relation

$$\hat{A}^+ = CA^T + \varepsilon [CB^T - CDCA^T], \tag{87}$$

where now the n -by- n matrices $C = (A^T A)^{-1}$ and $D = B^T A + A^T B$.

Also, $\hat{A}^+ \hat{A} = (\hat{A}^T \hat{A})^+ \hat{A}^T \hat{A} = (A^T A)^{-1} \hat{A}^T \hat{A} = I$. \square

Remark 8. If the matrix \hat{A}^+ exists, then the dual matrices $\hat{A}\hat{A}^+$, $\hat{A}^+ \hat{A}$, $I - \hat{A}\hat{A}^+$, and $I - \hat{A}^+ \hat{A}$ are symmetric and idempotent. Symmetry follows from the 3rd and 4th MP dual conditions. The first MP dual conditions yields $(\hat{A}\hat{A}^+) \hat{A}\hat{A}^+ = \hat{A}\hat{A}^+$, and hence idempotence. Likewise, the second MP dual condition yields $(\hat{A}^+ \hat{A}) \hat{A}^+ \hat{A} = \hat{A}^+ \hat{A}$.

Also, $(I - \hat{A}\hat{A}^+)(I - \hat{A}^+ \hat{A}) = I + \hat{A}\hat{A}^+ \hat{A}^+ \hat{A} - 2\hat{A}\hat{A}^+ = I - \hat{A}\hat{A}^+$. Idempotence can be likewise shown for $I - \hat{A}^+ \hat{A}$. Due to all their idempotences, these operators may be thought of as dual ‘projection matrices.’ \blacklozenge

Remark 9. [Result 8](#) depends on proving the identities in [Result 6](#) and the use of [Corollary 1](#).

It is useful to compare [Result 8](#) with recent results obtained in [Ref. \[13\]](#) for finding the MPDGI of an m -by- n dual matrix, $\hat{A} = A + \varepsilon B$, with $m \geq n$ in which A and B are restricted to have the same full rank. Formula (40) for the MPDGI of such a matrix, \hat{A} , in [Ref. \[13\]](#) requires the determination of the MP inverse of A and the MP inverse of $(A^T A)$, both of which would presumably be obtained computationally by using singular value decompositions (SVDs). In contrast: (i) [Eqs. \(78\)](#) and [\(79\)](#) do not restrict matrices A and B to have the same rank; (ii) these formulae solely require the determination of the regular inverse of a single symmetric full-rank matrix, which is conceptually simpler as well as computationally more efficient, and computationally more reliable; and (iii) more importantly, the MPDGI in [Result 8](#) is obtained directly in terms of the matrices A and B , the primal and dual parts of \hat{A} , with no SVD evaluations necessitated. \blacklozenge

As we shall see in the next section, when the rank, r , of the primal part, A , of a dual matrix \hat{A} is less than both the number of its columns and its rows, then there is no guarantee that the MPDGI of \hat{A} exists, and that, in fact, it almost always (generically) does not exist.

3. Use of dual generalized inverses for solving systems of linear dual equations

One of the main uses of dual equations in kinematics is the determination of the dual vector solution of matrix dual equations. Throughout this section we focus on the solution, \hat{x} , of the linear dual equation

$$\hat{A}\hat{x} = \hat{b} \tag{88}$$

in which \hat{A} is an m -by- n dual matrix and \hat{b} is an m -by-1 dual column vector whose elements are m dual numbers.

We begin in the next subsection by finding the condition for the set of dual equations given in Eq. (88) to be consistent. By consistent, we mean that Eq. (88) has at least one solution, \hat{x} . We start for now by assuming that the {1}-dual generalized inverse of the m -by- n dual matrix \hat{A} exists. We know that it always does when the primal part of \hat{A} has full column or full row rank (see the paragraph after Result 7). Later, in this subsection we consider in some detail when this condition is not true.

3.1. Equation $\hat{A}\hat{x} = \hat{b}$ is consistent and $\hat{A}^{(1)}$ exists

The {1}-dual generalized inverse of a dual matrix plays a central role in determining whether a set of linear dual equations has a solution, and if it does, in finding explicitly all the solutions to such a set. While we initially assume in this section that the {1}-dual inverse exists, we will later obtain the necessary and sufficient conditions for its existence. These conditions become important because if they are not met by a dual matrix its {1,3}-dual generalized inverse cannot exist nor can its Moore-Penrose dual generalized inverse. These two types of dual generalized inverses are shown in the following two sub-sections to be the dual analogs of the least squares generalized inverse and the least-squares minimum-norm generalized inverse for real matrices.

Result 9. If $\hat{A}^{(1)}$ exists, then the necessary and sufficient condition for the dual equation $\hat{A}\hat{x} = \hat{b}$ to be consistent is that

$$\hat{A}\hat{A}^{(1)}\hat{b} = \hat{b}. \tag{89}$$

Proof: Assume that $\hat{A}\hat{x} = \hat{b}$ is consistent; hence, this equation has a solution, \hat{x} . Pre-multiplying Eq. (88) on both sides by $\hat{A}\hat{A}^{(1)}$, we get

$$\hat{A}\hat{A}^{(1)}\hat{A}\hat{x} = \hat{A}\hat{A}^{(1)}\hat{b}. \tag{90}$$

But since $\hat{A}^{(1)}$ is assumed to exist, and it is the {1}-generalized dual inverse of \hat{A} , it satisfies the first MP dual condition given in Eq. (8), namely $\hat{A}\hat{A}^{(1)}\hat{A} = \hat{A}$. Hence Eq. (90) yields

$$\hat{A}\hat{A}^{(1)}\hat{A}\hat{x} = \hat{A}\hat{x} = \hat{b} = \hat{A}\hat{A}^{(1)}\hat{b}.$$

The last equality is therefore satisfied when the equation $\hat{A}\hat{x} = \hat{b}$ is consistent.

Now assume that $\hat{A}\hat{A}^{(1)}\hat{b} = \hat{b}$. Then by calling $\hat{x} = (\hat{A}^{(1)}\hat{b})$, we get $\hat{A}(\hat{A}^{(1)}\hat{b}) = \hat{A}\hat{x} = \hat{b}$. Hence \hat{x} is a solution of Eq. (88), and so the equation is consistent. \square

Since Eq. (89) gives the necessary and sufficient condition for a system of dual equations $\hat{A}\hat{x} = \hat{b}$ to be consistent, one can intuitively say that the extent, $\hat{\alpha}$, to which a given system of dual equations is *inconsistent* can be measured by the difference between the two sides of Eq. (89) so that $\hat{\alpha} = \hat{A}\hat{A}^{(1)}\hat{b} - \hat{b}$, and one succinct measure of this inconsistency is the norm of this dual vector, $(\hat{\alpha})$ (see Eq. (2)).

Corollary 2. If the matrix \hat{A}^+ exists, then the Eq. $\hat{A}\hat{x} = \hat{b}$ is consistent if and only if $\hat{A}\hat{A}^+\hat{b} = \hat{b}$.

Proof: Since $\hat{A}^+ = \hat{A}^{(1,2,3,4)}$, \hat{A}^+ is also a {1}-dual generalized inverse, and hence the result. \square

Result 10. If $\hat{A}^{(1)}$ exists, the general solution to the consistent equation $\hat{A}\hat{x} = \hat{b}$ is

$$\hat{x} = \hat{A}^{(1)}\hat{b} + (I - \hat{A}^{(1)}\hat{A})\hat{w}, \tag{91}$$

where \hat{w} is an arbitrary n -by-1 dual column vector.

Every solution to the system of dual equations can be written in the form of Eq. (91) for some n -by-1 dual vector \hat{w} .

Proof: Pre-multiplying Eq. (91) by \hat{A} we get

$$\hat{A}\hat{x} = \hat{A}\hat{A}^{(1)}\hat{b} + \hat{A}(I - \hat{A}^{(1)}\hat{A})\hat{w} = \hat{b}. \tag{92}$$

Since $\hat{A}\hat{x} = \hat{b}$ is consistent, by Result 9, the first member on the right-hand side of the first equality equals \hat{b} . And since $\hat{A}^{(1)}$ satisfies the relation $\hat{A}\hat{A}^{(1)}\hat{A} = \hat{A}$, the second member on the right-hand side of the first equality is zero. Hence the column vector \hat{x} given in Eq. (91) satisfies the equation $\hat{A}\hat{x} = \hat{b}$.

Also, every solution of the consistent equation $\hat{A}\hat{x} = \hat{b}$ can be put in the form given in Eq. (91). Since \hat{x} is a solution of the equation, pre-multiplying both sides of the equation by $\hat{A}^{(1)}$ we get

$$\hat{A}^{(1)}\hat{A}\hat{x} = \hat{A}^{(1)}\hat{b}, \tag{93}$$

or

$$0 = \hat{A}^{(1)}\hat{b} - \hat{A}^{(1)}\hat{A}\hat{x}. \tag{94}$$

Adding the vector \hat{x} on both sides gives

$$\hat{x} = \hat{A}^{(1)} \hat{b} + (I - \hat{A}^{(1)} \hat{A}) \hat{x} \tag{95}$$

which has the same form as Eq. (91) with $w = \hat{x}$. □

Result 10 says that the general equation $\hat{A}\hat{x} = \hat{b}$ in which \hat{A} is an m -by- n dual matrix does *not*, in general, have a unique solution. Note that given the matrix \hat{A} , its $\{1\}$ -dual inverse is not, in general, unique. The following example illustrates some of these results.

Example 4. Consider the solution of the dual equation $\hat{A}\hat{x} = \hat{b}$ where $\hat{A} = A + \varepsilon B$, where the matrices A and B are

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 7 & 6 & 3 & 1 \\ 5 & 4 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \end{bmatrix},$$

and $\hat{b} = c + \varepsilon d$ with $c = [12, 14, 19]^T$ and $d = [6, 37, 26]^T$.

Note that the 3-by-4 matrix A , which is the primal part of the dual matrix \hat{A} , has full row rank. By Results 7 and 8 it is therefore guaranteed to have a $\{1\}$ -dual generalized inverse.

A $\{1\}$ -dual generalized inverse, $\hat{A}^{(1)} = A^{(1)} + \varepsilon R$, of this matrix \hat{A} can be found that satisfies the first MP dual condition (8) (or alternatively Eq. (12)). $\hat{A}^{(1)}$ is not unique. One such $\{1\}$ -dual generalized inverse, shown to only 4 significant numbers, is given by (see Remark 7)

$$A^{(1)} = \begin{bmatrix} -0.1939 & -0.0706 & 0.2928 \\ 0.0617 & 0.1491 & -0.1491 \\ 0.3199 & 0.2916 & -0.5139 \\ 0.0277 & -0.2756 & 0.3867 \end{bmatrix}, \text{ and} \tag{96}$$

$$R = \begin{bmatrix} 0.138062 - 0.622222p_1, & 0.137148 - 0.622222p_2, & -0.312457 - 0.622222p_3 \\ p_1, & p_2, & p_3 \\ -0.577778p_1 - 0.763769, & -0.577778p_2 - 0.56594, & 1.14125 - 0.577778p_3 \\ 0.0888889p_1 + 0.288311, & 0.0888889p_2 + 0.390168, & 0.0888889p_3 - 0.777822 \end{bmatrix} \tag{97}$$

where $p_i, i = 1, 2, 3$, are arbitrary real numbers. Note that a matrix $A^+ := A^{(1,2,3,4)}$ is also an $A^{(1)}$.

A particular choice of $p_1 = p_2 = 1$, and $p_3 = 2$, in Eq. (97) yields, (see Remark 7)

$$R_1 = \begin{bmatrix} -0.4842 & -0.4851 & -1.5569 \\ 1.0000 & 1.0000 & 2.0000 \\ -1.3415 & -1.1437 & -0.0143 \\ 0.3772 & 0.4791 & -0.6000 \end{bmatrix}. \tag{98}$$

A different choice of $p_1 = 1, p_2 = 2, p_3 = 3$, in Eq. (97) yields,

$$R_2 = \begin{bmatrix} -0.4842 & -1.1073 & -2.1791 \\ 1.0000 & 2.0000 & 3.0000 \\ -1.3415 & -1.7215 & -0.5921 \\ 0.3772 & 0.5679 & -0.5112 \end{bmatrix}. \tag{99}$$

Using Eqs. (96)–(99) we then have two different $\{1\}$ -dual generalized inverses, $\hat{A}^{(1)} = A^{(1)} + \varepsilon R_i, i = 1, 2$. Clearly, there are an infinite number of them depending on the choice of the parameters $p_i, i = 1, 2, 3$.

To check that the dual system of linear equations is consistent, we first check that $\hat{A}A^{(1)}\hat{b} - \hat{b} = 0$, which it is. We now solve the dual equation using its explicit solution given in Eq. (91).

The first member on the right-hand side of Eq. (91) using Eq. (98) computes to

$$A^{(1)}\hat{b} = \begin{bmatrix} 2.2476 \\ -0.0051 \\ -1.8415 \\ 3.8218 \end{bmatrix} - \varepsilon \begin{bmatrix} 38.3440 \\ -66.0103 \\ 33.0328 \\ 0.1445 \end{bmatrix} \tag{100}$$

and the second member on the right computes to (see Remark 7)

$$(I - \hat{A}^{(1)}\hat{A})\hat{w} = \left(\begin{bmatrix} 0.2239 & -0.3599 & 0.2079 & -0.0320 \\ -0.3599 & 0.5784 & -0.3342 & 0.0514 \\ 0.2079 & -0.3342 & 0.1931 & -0.0297 \\ -0.0320 & 0.0514 & -0.0297 & 0.0046 \end{bmatrix} + \varepsilon \begin{bmatrix} 11.1209 & 9.9793 & 6.5928 & 6.9088 \\ -17.9126 & -15.9743 & -10.6324 & -11.0977 \\ 10.0575 & 9.6989 & 5.8720 & 6.4537 \\ -1.3671 & -1.7818 & -0.7360 & -1.0186 \end{bmatrix} \right) \hat{w} \tag{101}$$

where \hat{w} is any arbitrary 4-by-1 dual vector. ☹

Corollary 3. If a matrix \hat{A}^+ exists, the general solution to the consistent equation $\hat{A}\hat{x} = \hat{b}$ is

$$\hat{x} = \hat{A}^+\hat{b} + (I - \hat{A}^+\hat{A})\hat{w}, \tag{102}$$

where \hat{w} is an arbitrary n -by-1 dual vector.

Every solution to the system of dual equations can be written in the form of Eq. (102) for some n -by-1 dual vector \hat{w} .

Proof: A Moore-Penrose dual generalized inverse is also a {1}-dual generalized inverse, and hence the result.

Furthermore, if a unique solution exists for this system of dual equations then this solution must be $\hat{x} = \hat{A}^+\hat{b}$, and this can happen if and only if $I - \hat{A}^+\hat{A} = 0$, since \hat{w} is arbitrary. When the primal part A of the dual matrix \hat{A} has rank n , then $A^T A$ is a nonsingular n by n matrix, and so if \hat{A}^+ exists, (see Result 8, part (ii)) $\hat{A}^+\hat{A} = (\hat{A}^T \hat{A})^+ \hat{A}^T \hat{A} = (\hat{A}^T \hat{A})^{-1} (\hat{A}^T \hat{A}) = I$. The solution \hat{x} is then unique. \square

Corollary 4. If a matrix \hat{A}^+ exists, the solution to the homogeneous dual equation $\hat{A}\hat{x} = 0$ is

$$\hat{x} = (I - \hat{A}^+\hat{A})\hat{w}$$

where \hat{w} is an arbitrary n -by-1 dual column vector.

Proof: Set $\hat{b} = 0$ in Eq. (102). Thus the ‘null space’—all dual vectors \hat{x} such that $\hat{A}\hat{x} = 0$ —of the dual matrix \hat{A} is $I - \hat{A}^+\hat{A}$. \square

Result 10 gives the general solution of the equation $\hat{A}\hat{x} = \hat{b}$, if $A^{(1)}$ exists, and so it behooves us to find out the necessary and sufficient conditions for the {1}-dual generalized inverse of a dual matrix to exist. Recall that we have already proved in Result 7 (see Eqs. (66) and (67)) that if the primal part, A , of the m -by- n dual matrix $\hat{A} = A + \varepsilon B$ has either (i) full column rank, or (ii) full row rank, or (iii) is square and nonsingular, the matrix $\hat{A}^{(1)}$ always exists. Hence we need to consider dual matrices whose primal parts have rank $r < m, n$. We also know from Remark 6 that if the dual matrix \hat{A} has a {1}-dual inverse it must have the form

$$\hat{A}^{(1)} = A^{(1)} + \varepsilon R.$$

We now show that while all real matrices have {1}-generalized inverse, *not* all dual matrices \hat{A} have {1}-dual generalized inverses. We obtain the necessary and sufficient conditions for a dual matrix \hat{A} to have a {1}-dual generalized inverse.

Result 11. Consider the m -by- n dual matrix $\hat{A} = A + \varepsilon B$, whose primal part has rank $r < m, n$ and let the singular value decomposition of its primal part, A , be

$$A = \underbrace{U}_{m \times m} \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} \underbrace{V^T}_{n \times n} := \left[\begin{array}{c|c} U_1 & U_2 \\ \hline m \times r & m \times (m-r) \end{array} \right] \begin{bmatrix} \overbrace{\Lambda}^{r \times r} & \overbrace{0}^{r \times (n-r)} \\ \underbrace{0}_{(m-r) \times r} & \underbrace{0}_{(m-r) \times (n-r)} \end{bmatrix} \begin{bmatrix} \overbrace{V_1^T}^{r \times n} \\ \underbrace{V_2^T}_{(n-r) \times n} \end{bmatrix}, \tag{103}$$

where U and V are m -by- m and n -by- n orthogonal matrices, respectively, and $\Lambda > 0$ is the diagonal r -by- r matrix containing the singular values of the A . The matrix $\hat{A}^{(1)}$ exists if and only if the $(m-r)$ -by- $(n-r)$ matrix

$$W := U_2^T B V_2 = 0. \tag{104}$$

Proof: $\hat{A}^{(1)} = G + \varepsilon R := A^{(1)} + \varepsilon R$ and the n -by- m matrix $A^{(1)}$ can always be written as [18,19]

$$G := A^{(1)} = V \begin{bmatrix} \overbrace{\Lambda^{-1}}^{r \times r} & \overbrace{K}^{r \times (m-r)} \\ \underbrace{L}_{(n-r) \times r} & \underbrace{M}_{(n-r) \times (m-r)} \end{bmatrix} U^T \tag{105}$$

in which the matrices K , L , and M have the appropriate dimensions shown, and are arbitrary. It can be directly checked using Eqs. (103) and (105) that $AA^{(1)}A = A$. Notice that $A^{(1)}$ is obviously not unique since the three matrices K , L , and M , have arbitrary elements in them.

Assume that $\hat{A}^{(1)} = A^{(1)} + \varepsilon R$ exists so that the 1st MP dual condition given in Eq. (12), namely

$$B = BA^{(1)}A + ARA + AA^{(1)}B \tag{106}$$

is satisfied by the matrix R . On using Eqs. (103) and (104), this relation reduces to

$$B = BV \begin{bmatrix} I & 0 \\ L\Lambda & 0 \end{bmatrix} V^T + U \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} V^T + U \begin{bmatrix} I & \Lambda K \\ 0 & 0 \end{bmatrix} U^T B \tag{107}$$

where $X := \Lambda V_1^T R U_1 \Lambda$. Post-multiplying both sides of Eq. (107) by the orthogonal matrix V and pre-multiplying both sides by the orthogonal matrix U , gives

$$U^T B V = U^T B V \begin{bmatrix} I & 0 \\ L\Lambda & 0 \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} I & \Lambda K \\ 0 & 0 \end{bmatrix} U^T B V. \tag{108}$$

Since

$$U^T B V = \begin{bmatrix} U_1^T B \\ U_2^T B \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} = \begin{bmatrix} \underbrace{U_1^T B V_1}_P & \underbrace{U_1^T B V_2}_Q \\ \underbrace{U_2^T B V_1}_S & \underbrace{U_2^T B V_2}_W \end{bmatrix} := \begin{bmatrix} \underbrace{r \times r}_P & \underbrace{r \times (n-r)}_Q \\ \underbrace{(m-r) \times r}_S & \underbrace{(m-r) \times (n-r)}_W \end{bmatrix}, \tag{109}$$

using Eq. (109), Eq. (108) reduces to

$$\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -Y & -\Lambda K W \\ -W \Lambda & W \end{bmatrix}, \tag{110}$$

where the r -by- r matrix $Y = P + Q \Lambda + \Lambda K S$. Note that Eq. (110) is equivalent to Eq. (106), which is the 1st MP-dual condition.

Equating the block matrices in Eq. (110) we get

$$W = U_2^T B V_2 = 0, \tag{111}$$

and

$$X = \Lambda V_1^T R U_1 \Lambda = -Y = -(P + Q \Lambda + \Lambda K S). \tag{112}$$

We have thus shown that if $\hat{A}^{(1)}$ exists then this implies that $W = U_2^T B V_2 = 0$.

We now prove the converse. Assume that $W = U_2^T B V_2 = 0$, we must show that $\hat{A}^{(1)} = A^{(1)} + \varepsilon R$ exists. $A^{(1)}$ always exists, since every real matrix has a $\{1\}$ -generalized inverse, so we must show that we can always find a matrix R that satisfies Eq. (112).

Pre- and post-multiplying both sides of the last equality in Eq. (112) by Λ^{-1} gives the r -by- r matrix equation

$$V_1^T R U_1 = -\Lambda^{-1} Y \Lambda^{-1} = -(\Lambda^{-1} P \Lambda^{-1} + \Lambda^{-1} Q \Lambda + K S \Lambda^{-1}) := \bar{Y} \tag{113}$$

and we must show this equation is consistent.

To show this, we use the necessary and sufficient condition for its consistency, which is that $V_1^T (V_1^T)^+ \bar{Y} U_1 + U_1 = \bar{Y}$ [18]. But $V_1^+ = V_1^T$ since the columns of V_1 are orthogonal; likewise, $U_1^+ = U_1^T$. Hence, $V_1^T (V_1^T)^+ = V_1^T (V_1^+)^T = V_1^T V_1 = I_r$, and likewise, $U_1^+ U_1 = U_1^T U_1 = I_r$. Hence the consistency condition is satisfied. This proves the converse.

In fact, we can find one solution of Eq. (113) quite handily. To do this, we posit the solution $R = \hat{R} = V_1 Z U_1^T$. This matrix \hat{R} always satisfies Eq. (113) because

$$V_1^T \hat{R} U_1 = \underbrace{V_1^T V_1}_=I_r Z \underbrace{U_1^T U_1}_=I_r = -(\Lambda^{-1} P \Lambda^{-1} + \Lambda^{-1} Q \Lambda + K S \Lambda^{-1}), \tag{114}$$

which yields $Z = -(\Lambda^{-1} P \Lambda^{-1} + \Lambda^{-1} Q \Lambda + K S \Lambda^{-1})$. Hence, one solution of Eq. (113) is

$$R = \hat{R} = -V_1 (\Lambda^{-1} P \Lambda^{-1} + \Lambda^{-1} Q \Lambda + K S \Lambda^{-1}) U_1^T. \tag{115}$$

□

Remark 10. Result 11 states that for an m -by- n dual matrix $\hat{A} = A + \varepsilon B$ in which the matrix A has rank $r < m, n$ to have an $\{1\}$ -dual generalized inverse, the matrix B cannot be chosen independently of the matrix A ! To see this, recall the singular value decomposition $A = U_1 \Lambda V_1^T$ given in Eq. (103). Since V is an orthogonal matrix, the subspace spanned by the columns of V_2 is orthogonal to the sub-space spanned by the columns of V_1 . Also, since U is orthogonal, the subspace orthogonal to that spanned by the column vectors of U_2 is the sub-space spanned by the column vectors of U_1 .

The relation $W = U_2^T B V_2 = 0$ in Eq. (111) places a constraint on the elements of the matrix B . Since W is an $(m-r)$ -by- $(n-r)$ matrix, it requires that $(m-r) \times (n-r)$ conditions to be exactly satisfied. It says that the m -by- n matrix B is required to map all n -by-1 column vectors that belong to the vector sub-space spanned by the columns of V_2 —which is the vector sub-space orthogonal to that spanned by the columns of V_1 — so that these mapped m -by-1 vectors are orthogonal to the vector sub-space spanned by the columns of U_2 , —which is the sub-space spanned by the columns of U_1 . In short, B must map all n -by-1 column vectors that are orthogonal to the sub-space spanned by the columns of V_1 to m -by-1 column vectors that belong to the sub-space spanned by the columns of U_1 ; and U_1 and V_1 come from the real matrix A 's singular value decomposition. Then, and only then, can a $\{1\}$ -dual generalized inverse of the dual matrix $\hat{A} = A + \varepsilon B$ exist.

To summarize, when the primal part, A , of the m -by- n dual matrix $\hat{A} = A + \varepsilon B$ has:

- (i) rank $r = m$, then $U_2 = 0$, so that relation $W = U_2^T B V_2 = 0$ is automatically satisfied, and no constraint needs to be placed on the elements of the matrix B for an $\hat{A}^{(1)}$ to exist. In fact, Result 8 indeed shows that (a unique) $A^{(1,2,3,4)}$ exists, and therefore the existence of an $\hat{A}^{(1)}$ is assured;
- (ii) rank $r = n$, then $V_2 = 0$, so that the relation $U_2^T B V_2 = 0$ is again automatically satisfied, and again no constraint needs to be placed on the elements of the matrix B for an $\hat{A}^{(1)}$ to exist. Again, Result 8 shows that $A^{(1,2,3,4)}$ exists, and therefore an $\hat{A}^{(1)}$ is assured to exist;

(iii) rank $r < m, n$, then the matrix B , which is the dual part of \hat{A} , cannot be chosen independently from the matrix $A = U_1 \Lambda V_1^T$, which is the primal part of \hat{A} , in order for \hat{A} to have an $\hat{A}^{(1)}$. This is because now B must satisfy Eq. (104). Thus, the matrix B must be a linear transformation such that for every column vector v that is orthogonal to the column space of V_1 , the column vector Bv must belong to the column space of U_1 . It is therefore, indeed, generic for a dual matrix \hat{A} whose primal part has rank $r < m, n$ to have no $\{1\}$ -dual generalized inverse. That is, almost all matrices with rank $r < m, n$ do not have $\{1\}$ -dual generalized inverses, and therefore also no $\{1,3\}$ -dual generalized inverses and no MPDGs. ♦

It should be noted from item (iii) above that m -by- n dual matrices \hat{A} that have primal parts that are rank deficient (have rank $r < m, n$) generically do not have $\{1\}$ -dual generalized inverses. This is in stark contrast to the situation with real m -by- n matrices that are guaranteed to have $\{1\}$ -generalized inverses no matter what their rank.

Result 11 is a generalization of the result previously obtained in Ref. [17], which shows that not all dual matrices have MPDGs. There, a proof was given by construction showing that there exist an uncountably infinite number of dual matrices that have no $\{1\}$ -dual generalized inverses [17]. Here we have shown that a necessary and sufficient condition for a dual matrix to have a $\{1\}$ -dual generalized inverse is that $U_2^T B V_2 = 0$. Thus a necessary condition for a dual matrix to have a $\{*\}$ -dual generalized inverse in which $\{*\}$ contains one or more distinct elements of the set $\{1,2,3,4\}$ one of which is *always* 1, is that $U_2^T B V_2 = 0$. This condition is almost always (generically) not satisfied by dual matrices whose primal parts are rank deficient. Hence, such dual matrices almost always have no MPDGs. As seen in hindsight from the constructive proof given in Ref. [17], the construction there begins (now quite predictably) with dual matrices whose primal parts are rank deficient.

Furthermore, since the matrices K, L , and M in Eq. (105) are arbitrary, one can choose them all to be zero matrices. Then $A^{(1)} = A^+$ and from Eqs. (115) and (109) we obtain

$$R = -\underbrace{V_1 \Lambda^{-1} U_1^T}_{A^+} B V_1 \Lambda^{-1} U_1^T = -A^+ B A^+.$$

Thus, a $\{1\}$ -dual generalized inverse of any m -by- n matrix $\hat{A} = A + \varepsilon B$ in which A is rank deficient with $U_2^T B V_2 = 0$ is given by

$$\hat{A}^{(1)} = A^+ - \varepsilon A^+ B A^+. \tag{116}$$

It is important to note that while Eq. (116) looks the same as the one obtained in Result 7, it applies to a very different kind of dual matrix \hat{A} from that given in Result 7. Eq. (116) applies to a dual matrix \hat{A} : (i) whose primal part is rank deficient, and (ii) whose dual part satisfies the necessary and sufficient condition $U_2^T B V_2 = 0$. Furthermore, this $\{1\}$ -dual generalized inverse is guaranteed to satisfy *only* the first MP dual condition.

Corollary 5. The m -by- n dual matrix $\hat{A} = A + \varepsilon B$ in which (i) the rank, r , of A is less than m and n , and (ii) $U_2^T B V_2 = 0$, has a $\{1\}$ -dual generalized inverse given explicitly by $\hat{A}^{(1)} = A^{(1)} + \varepsilon R$ in which the matrix R must satisfy Eq. (113). The matrices V_1 and U_1 come from the singular value decomposition $A = U_1 \Lambda V_1^T$ in which Λ is an r -by- r diagonal positive definite matrix, the matrices P, Q, R are defined in Eq. (109), and the matrices L and K are arbitrary. The elements of the matrix R are not unique.

Proof: We note that the matrix $V_1^T R U_1$ is an r -by- r matrix and has r^2 elements, so that this matrix equation yields for a given choice of matrices L, K , and M , a set of r^2 equations for determining the $mn > r^2$ elements of the matrix R , since $r < m, n$. Hence, we have an under-determined set of linear equations with more unknown elements of the matrix R than we have equations and the determination of the elements of the matrix R cannot be unique. One such specific $\{1\}$ -dual inverse is provided by Eq. (116). □

We therefore see that when the primal part of an m -by- n dual matrix has full row or full column rank, then the matrix has an MPDGI, and therefore a $\{1\}$ -dual generalized inverse. But when the primal part of a dual matrix has rank $r < m, n$, it generically (most likely, in general) does not have a $\{1\}$ -dual generalized inverse. If it does, then $U_2^T B V_2 = 0$, and furthermore, the dual part, R , of the $\{1\}$ -dual inverse of such a dual matrix is not unique. The next example illustrates this.

Example 5. Consider the 3-by-3 dual matrix

$$\hat{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix} + \varepsilon \begin{bmatrix} b_1 & b_4 & b_7 \\ b_2 & b_5 & b_8 \\ b_3 & b_6 & b_9 \end{bmatrix} := A + \varepsilon B \tag{117}$$

Our aim is to see under what circumstances \hat{A} has a $\{1\}$ -dual generalized inverse, and then to find it. We have $\hat{A}^{(1)} = A^{(1)} + \varepsilon R$, when it exists.

The rank of the real part, A , of \hat{A} is 2. The singular value decomposition of A is (see Remark 7)

$$A = \underbrace{\begin{bmatrix} -0.4082 & 0.7071 & -0.5774 \\ -0.4082 & -0.7071 & -0.5774 \\ -0.8165 & -0.0000 & 0.5774 \end{bmatrix}}_U \underbrace{\begin{bmatrix} 5.7446 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_\Lambda \underbrace{\begin{bmatrix} -0.6396 & -0.7071 & 0.3015 \\ -0.6396 & 0.7071 & 0.3015 \\ -0.4264 & 0 & -0.9045 \end{bmatrix}^T}_V \tag{118}$$

so that $U_2 = 0.5774[-1, -1, 1]^T$ and $V_2 = 0.315[1, 1, -3]^T$. According to Result 11, the dual matrix \hat{A} given in Eq. (117) can have a {1}-dual generalized inverse if and only if $U_2^T B V_2 = 0$. This condition requires that the elements of the matrix B satisfy the relation

$$(1/3)(-b_1 - b_2 + b_3 - b_4 - b_5 + b_6) + b_7 + b_8 = b_9 \tag{119}$$

Thus, for the matrix $A^{(1)}$ to exist, all the elements of the matrix B in Eq. (117) can be arbitrary *except* the element b_9 , which must satisfy Eq. (119) *exactly*.

The fact that the elements of the matrix B have to satisfy condition (119) *exactly* for the existence of a {1}-inverse shows that it is generic (most likely, in general) for the dual matrix \hat{A} in Eq. (117), in which the matrix B has arbitrary elements, to have *no* {1}-dual generalized inverse.

For example, when $b_i, i = 1, 2, \dots, 8$, Eq. (119) informs us that we *must* have $b_9 = 14$ in order for the dual matrix

$$\hat{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & b_9 \end{bmatrix} := A + \varepsilon B. \tag{120}$$

to have a {1}-dual generalized inverse. Every number other 14 will cause the matrix in Eq. (120) to have no $\hat{A}^{(1)}$. Assuming that $b_9 = 14$, the slightest alteration in *any* of the other elements of the matrix B above will most likely again cause the {1}-dual inverse of the dual matrix \hat{A} not to exist, because then Eq. (119) will not be satisfied. This is what we mean when we say that generically the matrix \hat{A} in Eq. (117) has no {1}-dual generalized inverse.

We also now see why the matrix \hat{A} in Example 1 (Eq. (23)) has no {1}-dual generalized inverse because the (3,3) element of its dual part is different from 14.

To find $\hat{A}^{(1)}$ of the matrix \hat{A} in Eq. (120), we begin by finding the {1}-inverse of A , its primal part. Using Eq. (105), with the arbitrary matrices K, L , and M chosen as $K = [3 \ 3]^T, L = [2 \ 1]$, and $M = 3$, we get a {1}-inverse of A as (see Remark 7)

$$A^{(1)} = \begin{bmatrix} 1.3228 & 1.8964 & -2.2118 \\ -0.1267 & -1.5531 & 0.2377 \\ 2.4345 & 3.7137 & -0.7675 \end{bmatrix}. \tag{121}$$

Note that since the matrices K, L , and M are arbitrary, $\hat{A}^{(1)}$ is not unique.

To determine the dual part of $\hat{A}^{(1)}$, we use the singular value decomposition of A to find U_1 and V_1 (Eq. (118)), and then solve the linear set of four equations given in Eq. (113) to get (see Remark 7)

$$R = \begin{bmatrix} p_2 - p_1/3 + p_4/3 - 5.8445, & p_2, & -p_2 - p_4/3 - p_5/3 - 3.4247 \\ p_3 - p_1/3 + p_6/3 + 1.0776, & p_3, & -p_3 - p_4/3 - p_5/3 - 4.5588 \\ p_1, & p_4, & p_5 \end{bmatrix}, \tag{122}$$

where the real numbers $p_i, i = 1, \dots, 5$ are arbitrary. As expected, the matrix R , is not unique. The matrix \hat{A} given in Eq. (120) then has the non-unique {1}-dual generalized inverse

$$\hat{A}^{(1)} = A^{(1)} + \varepsilon R \tag{123}$$

where $\hat{A}^{(1)}$ and R are given in Eqs. (121) and (122) respectively; the real parameters $p_i, i = 1, \dots, 5$ are arbitrary in R .

Choosing $p_1 = p_4 = 2, p_2 = 3, p_3 = 1$, and $p_5 = 7$, we get a specific {1}-dual generalized inverse of the dual matrix \hat{A} in Eq. (120) given by (see Remark 7)

$$\hat{A}^{(1)} = \begin{bmatrix} 1.3228 & 1.8964 & -2.2118 \\ -0.1267 & -1.5531 & 0.2377 \\ 2.4345 & 3.7137 & -0.7675 \end{bmatrix} + \varepsilon \begin{bmatrix} -2.8445 & 3.0000 & -9.4247 \\ 2.0776 & 1.0000 & -8.5588 \\ 2.0000 & 2.0000 & 7.0000 \end{bmatrix}. \tag{124}$$

⌘

Example 6. Notice that the matrix \hat{A} in Eq. (117) can be expressed in another basis set. That will not change the fact that generically speaking the new matrix obtained will generically have no {1}-dual generalized inverse. For example, in the non-orthogonal basis set (see Result 2, and Remarks 3 and 8)

$$T = \begin{bmatrix} 0.4082 & 0.7071 & -0.3015 \\ 0.4082 & -0.7071 & -0.3015 \\ 0.8165 & -0.0000 & 0.9045 \end{bmatrix} \tag{125}$$

the matrix representation of this dual matrix \hat{A} is

$$\tilde{\hat{A}} = T^{-1}(A + \varepsilon B)T = \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\hat{A}} + \varepsilon T^{-1}BT = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} b'_1 & b'_4 & b'_7 \\ b'_2 & b'_5 & b'_8 \\ b'_3 & b'_6 & b'_9 \end{bmatrix}. \tag{126}$$

Since the primal part, \tilde{A} , of the dual matrix $\tilde{\hat{A}}$ is much simpler than A in Eq. (117) it is clearly easier to work with. In fact, the singular value decomposition of \tilde{A} is trivial now. It is simply

$$\tilde{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_U \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_V^T,$$

so that the matrices $U_2 = V_2 = [0 \ 0 \ 1]^T$.

The matrix $\tilde{\hat{A}}$ has a {1}-dual generalized inverse if and only if $U_2^T B V_2 = 0$. This implies that $b'_9 = 0$, with $b'_i, i = 1, 2, \dots, 8$, arbitrary. Taking $b'_i = i, i = 1, 2, \dots, 8$, we find that the dual matrix

$$\tilde{\hat{A}} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 0 \end{bmatrix} \tag{127}$$

has a {1}-dual generalized inverse,

$$\tilde{\hat{A}}^{(1)} = \begin{bmatrix} 0.2 & 0 & 2 \\ 0 & -1 & -3 \\ 2 & 2 & 3 \end{bmatrix} + \varepsilon \begin{bmatrix} -4.04 & 10 & p_3 \\ 18.2 & -7.0 & p_4 \\ p_1 & p_2 & p_5 \end{bmatrix}, \tag{128}$$

where $p_i, i = 1, \dots, 5$ are arbitrary. The arbitrary matrices (see Eq. (105)) are taken to be $K = [2 \ 3]^T, L = [2 \ 2]$, and $M = 3$ when finding the primal part, $A^{(1)}$, of $\tilde{\hat{A}}^{(1)}$ (see Eq. (105)).

A specific {1}-dual inverse with $p_1 = p_4 = 2, p_2 = 3, p_3 = 1$, and $p_5 = 7$, is then

$$\tilde{\hat{A}}^{(1)} = \begin{bmatrix} 0.2 & 0 & 2 \\ 0 & -1 & -3 \\ 2 & 2 & 3 \end{bmatrix} + \varepsilon \begin{bmatrix} -4.04 & 10 & 1 \\ 18.2 & -7.0 & 2 \\ 2 & 3 & 7 \end{bmatrix}. \tag{129}$$

As seen, the computational burden in finding $\tilde{\hat{A}}^{(1)}$ and the accuracy of the result is greatly enhanced by using the transformation T .

Using the transformation T , one can now go back to the original standard coordinate basis to obtain a dual matrix that has a {1}-dual generalized inverse (see Remark 7)

$$\hat{C} = T \tilde{\hat{A}} T^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix} + \varepsilon \begin{bmatrix} -2.5979 & -7.3489 & 6.0976 \\ -2.1717 & 3.0773 & -2.7927 \\ 4.9499 & -7.3441 & 5.5206 \end{bmatrix} \tag{130}$$

which, by Result 2 and Remark 2 is guaranteed to have a {1}-dual generalized inverse, which is obtained by using the transformation T to express $\tilde{\hat{A}}^{(1)}$ in the original coordinates so that

$$\hat{C}^{(1)} = T \tilde{\hat{A}}^{(1)} T^{-1} = \begin{bmatrix} 0.1560 & 2.0088 & -1.7209 \\ -1.6583 & -1.8055 & 1.0933 \\ -0.1546 & -2.7130 & 3.8495 \end{bmatrix} + \varepsilon \begin{bmatrix} 6.7401 & 9.2458 & 5.0101 \\ -3.2977 & -14.7920 & -9.4754 \\ 1.8561 & -13.5285 & 4.0119 \end{bmatrix}$$

It is easy to check that $\hat{C}^{(1)}$ is the {1}-dual generalized inverse of \hat{C} . Having found one specific {1}-inverse, others can be found using the following corollary. ☼

We next show how one can generate other {1}-dual generalized inverses of a given dual matrix \hat{A} given one specific {1}-dual matrix, such as the ones found in Eqs. (124) and (129).

Corollary 6. Assuming that the {1}-dual generalized inverse of the dual m -by- n matrix \hat{A} exists, and that one such $\hat{A}^{(1)}$ is known, the set of matrices

$$\hat{S}(\hat{P}, \hat{Q}) = \hat{A}^{(1)} \hat{A} \hat{A}^{(1)} + (I_n - \hat{A}^{(1)} \hat{A}) \hat{P} + \hat{Q} (I_m - \hat{A} \hat{A}^{(1)}), \tag{131}$$

where \hat{P} and \hat{Q} are arbitrary n -by- m dual matrices, are all {1}-dual inverses of the matrix \hat{A} .

Proof: To show that every member of the set of matrices \hat{S} (whose value depends on the values chosen for arbitrary dual matrices \hat{P} and \hat{Q}) is a {1}-dual generalized inverses of \hat{A} , we pre- and post-multiply by the matrix \hat{A} to get

$$\hat{A} \hat{S} \hat{A} = \underbrace{\hat{A} \hat{A}^{(1)} \hat{A}}_{\hat{A}} \hat{A} \hat{A}^{(1)} \hat{A} + \underbrace{(\hat{A} - \hat{A} \hat{A}^{(1)} \hat{A})}_{0} \hat{P} \hat{A} + \hat{A} \hat{Q} \underbrace{(\hat{A} - \hat{A} \hat{A}^{(1)} \hat{A})}_{0} = \hat{A}, \tag{132}$$

showing that every member of the set \hat{S} is a $\{1\}$ -dual generalized inverse. \square

Remark 11. To get $\hat{A}^{(1)}$, of a dual matrix $\hat{A} = A + \varepsilon B$ when A has rank $r < m, n$ one first needs to find the matrix $A^{(1)}$, which is nonunique, in general. Then, $\hat{A}^{(1)} = A^{(1)} + \varepsilon R$. There is a simple way to solve Eq. (113) to get one specific matrix R as demonstrated in Eqs. (114) and (115) in Result 11 by positing the solution to be $R = \hat{R} := V_1 Y U_1^T$. Other $\{1\}$ -dual generalized inverses, if needed, can then be generated from this $\{1\}$ -dual generalized inverse by using Corollary 6. \blacklozenge

With these results we have a near-complete theory for the solution of the linear dual equation $\hat{A}\hat{x} = \hat{b}$, in which \hat{A} is an m -by- n matrix having covered the cases when the primal part, A , of \hat{A} has: full row rank, full column rank, and rank $r < m, n$.

In the following two subsections we shall assume that the m -by- n dual matrix $\hat{A} = A + \varepsilon B$ has a $\{1,2,3,4\}$ -dual generalized inverse and therefore also a $\{1, 3\}$ -dual generalized inverse. When A has either full row rank or full column rank, we know from Result 8 that a $\{1,2,3,4\}$ -dual generalized inverse (MPDGI) of \hat{A} always exists, and we can use the simple explicit expressions given in Eqs. (78) and (79) to find it. It is therefore only when $\text{rank}(A) = r < m, n$ that this assumption comes into play. We know from the previous section that such dual matrices do not generically have MPDGIs.

3.2. Analogue of the least-squares solution of the possibly inconsistent dual equation $\hat{A}\hat{x} = \hat{b}$

In this subsection we consider the problem of finding the dual vector \hat{x} that is analogous to the least-squares solution that minimizes the Euclidean norm of the error $e = Ax - b$ when the matrix A and the column vector b are real.

In the equation $\hat{A}\hat{x} = \hat{b}$, for any \hat{x} chosen the error is measured by $\hat{e} = A\hat{x} - b$, and its norm $\langle \hat{e} \rangle$.

Recall that the $\{1,3\}$ -dual generalized inverse of the matrix \hat{A} satisfies the two MP dual conditions (see Eqs. (8) and (10)),

$$\hat{A}\hat{A}^{\{1,3\}}\hat{A} = \hat{A}, \tag{133}$$

$$\hat{A}\hat{A}^{\{1,3\}} = (\hat{A}\hat{A}^{\{1,3\}})^T, \tag{134}$$

and that every $\{1,3\}$ -dual generalized inverse is also a $\{1\}$ -dual generalized inverse. Practical examples in application areas like kinematics and robotics in which least-squares type solutions of dual matrices are sought are common (see for example Ref. 14).

Corollary 7. If the m -by- n matrix $\hat{A} = A + \varepsilon B$ has an MPDGI, then

$$\hat{A}\hat{A}^{\{1,3\}} = \hat{A}\hat{A}^+ \tag{135}$$

Proof: We have

$$\begin{aligned} \hat{A}\hat{A}^{\{1,3\}} &= \hat{A}\hat{A}^+\hat{A}\hat{A}^{\{1,3\}} = (\hat{A}\hat{A}^+)^T(\hat{A}\hat{A}^{\{1,3\}})^T \\ &= (\hat{A}^+)^T \underbrace{\hat{A}^T(\hat{A}^{\{1,3\}})^T\hat{A}^T}_{(\hat{A}^+)^T\hat{A}^T} = (\hat{A}^+)^T\hat{A}^T = (\hat{A}\hat{A}^+)^T\hat{A}\hat{A}^+. \end{aligned} \tag{136}$$

\square

We have shown here that even though the $\{1,3\}$ -dual generalized inverse of a given dual matrix \hat{A} may not be unique, the product $\hat{A}\hat{A}^{\{1,3\}}$ is always unique, since \hat{A}^+ , which we assume in this sub-section always exists, is unique. Thus, no matter which specific $\{1,3\}$ -dual generalized inverse of \hat{A} is used, $\hat{A}\hat{A}^{\{1,3\}}$ has the same value.

Result 12. If the matrix \hat{A}^+ exists, then the dual vector

$$\hat{x} = \hat{A}^{\{1,3\}}\hat{b} + (I - \hat{A}^{\{1,3\}}\hat{A})\hat{w} \tag{137}$$

where \hat{w} is an arbitrary n -by-1 dual vector, is the solution to the possibly inconsistent equation $\hat{A}\hat{x} = \hat{b}$. The norm of the error

$$\langle \hat{e} \rangle = \langle \hat{A}\hat{x} - b \rangle = \langle (\hat{A}\hat{A}^{\{1,3\}} - I)\hat{b} \rangle = \langle (\hat{A}\hat{A}^+ - I)\hat{b} \rangle. \tag{138}$$

Furthermore,

$$\hat{A}\hat{x} = \hat{A}\hat{A}^{\{1,3\}}\hat{b}. \tag{139}$$

Proof: Adding and subtracting $\hat{A}\hat{A}^{\{1,3\}}\hat{b}$ we get

$$\hat{e} = A\hat{x} - \hat{b} = \underbrace{\hat{A}(\hat{x} - \hat{A}^{\{1,3\}}\hat{b})}_{\hat{u}_1} + \underbrace{(\hat{A}\hat{A}^{\{1,3\}}\hat{b} - \hat{b})}_{\hat{u}_2} := \underbrace{\hat{A}\hat{y}}_{\hat{u}_1} + \underbrace{(\hat{A}\hat{A}^{\{1,3\}}\hat{b} - \hat{b})}_{\hat{u}_2} \tag{140}$$

where we have denoted $(\hat{x} - \hat{A}^{\{1,3\}}\hat{b}) := \hat{y}$.

We next consider the inner product of the two members on the right-hand side of the last equality. We find that

$$\begin{aligned} \hat{u}_2^T\hat{u}_1 &= (\hat{A}\hat{A}^{\{1,3\}}\hat{b} - \hat{b})^T\hat{A}\hat{y} = \hat{b}^T(\hat{A}\hat{A}^{\{1,3\}} - I)^T\hat{A}\hat{y} = \hat{b}^T[(\hat{A}\hat{A}^{\{1,3\}})^T - I]\hat{A}\hat{y} \\ &= \hat{b}^T[\hat{A}\hat{A}^{\{1,3\}} - I]\hat{A}\hat{y} = \hat{b}^T[\hat{A}\hat{A}^{\{1,3\}}\hat{A} - \hat{A}]\hat{y} = 0. \end{aligned} \tag{141}$$

To get the first equality on the second line we have used Eq. (134), and to get the last equality we have used Eq. (133). From Eq. (140) and recalling the last equality in the first line in Eq. (3), the square of the Euclidean norm of \hat{e} yields

$$\begin{aligned} \|\hat{e}\|^2 &= \|\hat{A}\hat{x} - \hat{b}\|^2 = \|\hat{u}_1 + \hat{u}_2\|^2 = \|\hat{u}_1\|^2 + \|\hat{u}_2\|^2 + 2\varepsilon\hat{u}_2^T\hat{u}_1 = \|\hat{u}_1\|^2 + \|\hat{u}_2\|^2 \\ &= \left\| \underbrace{\hat{A}\hat{x} - \hat{A}\hat{A}^{(1,3)}\hat{b}}_{\hat{u}_1=p_1+\varepsilon q_1} \right\|^2 + \left\| \underbrace{(\hat{A}\hat{A}^{(1,3)} - I)\hat{b}}_{\hat{u}_2=p_2+\varepsilon q_2} \right\|^2. \end{aligned} \tag{142}$$

Denoting the dual vectors $\hat{u}_i = p_i + \varepsilon q_i$, $i = 1, 2$, shown in Eq. (140) and using Eq. (7), a bound on the norm of the error \hat{e} is given by

$$\langle \hat{e} \rangle \leq \langle \hat{e} \rangle_{bound} = \sqrt{\|p_1\|^2 + \|p_2\|^2 + \|q_1\| + \|q_2\|}, \tag{143}$$

in which the norms of the dual vectors \hat{u}_i are $\langle \hat{u}_i \rangle = \|p_i\| + \|q_i\|$, $i = 1, 2$. Eq. (143) informs us that this bound, $\langle \hat{e} \rangle_{bound}$, on the norm of the dual error \hat{e} contains the quantities $\|p_i\|$, $\|q_i\|$, $i = 1, 2$ that belong to the norm of the dual vectors \hat{u}_i , $i = 1, 2$. Notice that the vector \hat{u}_2 in Eq. (142) depends only on the matrix \hat{A} , and \hat{b} (and the {1,3}-inverse of \hat{A}). So in order to obtain the smallest value of $\langle \hat{e} \rangle_{bound}$ given in Eq. (143) we can choose our \hat{x} in Eq. (142) to make the dual vector $\hat{u}_1 = 0$, so that $\|p_1\| = \|q_1\| = 0$.

In fact, the choice of \hat{x} given in Eq. (137) does exactly this. It causes \hat{u}_1 in Eq. (142) to vanish, because

$$\begin{aligned} \hat{A}\hat{x} - \hat{A}\hat{A}^{(1,3)}\hat{b} &= \hat{A}[\hat{A}^{(1,3)}\hat{b} + (I - \hat{A}^{(1,3)}\hat{A})\hat{w}] - \hat{A}\hat{A}^{(1,3)}\hat{b} \\ &= \hat{A}\hat{A}^{(1,3)}\hat{b} + \hat{A}(I - \hat{A}^{(1,3)}\hat{A})\hat{w} - \hat{A}\hat{A}^{(1,3)}\hat{b} \\ &= \hat{A}\hat{A}^{(1,3)}\hat{b} + \underbrace{(\hat{A} - \hat{A}\hat{A}^{(1,3)}\hat{A})}_{=0}\hat{w} - \hat{A}\hat{A}^{(1,3)}\hat{b} = 0. \end{aligned} \tag{144}$$

This proves Eqs. (137) and (142) then yields

$$\|\hat{e}\|^2 = \|\hat{u}_2\|^2 = \left\| \underbrace{(\hat{A}\hat{A}^{(1,3)} - I)\hat{b}}_{\hat{u}_2=p_2+\varepsilon q_2} \right\|^2, \tag{145}$$

so that

$$\langle \hat{e} \rangle = \langle \hat{u}_2 \rangle = \|p_2\| + \|q_2\| = \langle \hat{A}\hat{A}^{(1,3)}\hat{b} - \hat{b} \rangle \tag{146}$$

When the system of dual equations $\hat{A}\hat{x} = \hat{b}$ is consistent then from Result 9 we see that $\langle \hat{e} \rangle = 0$, since a {1,3}-dual generalized inverse is also a {1}-dual generalized inverse. When the system is inconsistent then the 'extent' to which it is inconsistent is provided by the difference $\hat{u}_2 = \hat{A}\hat{A}^{(1,3)}\hat{b} - \hat{b}$ —see Result 9, Eq. (89)—, and from Eq. (146) we see that $\langle \hat{e} \rangle$ is, indeed, the norm of this quantity. While the solution \hat{x} given in Eq. (137) may not be unique, by Corollary 7 the value of the error norm $\langle \hat{e} \rangle$ in Eq. (146) is always the same since \hat{A}^+ is unique.

From Eq. (137) we obtain the relation $\hat{A}\hat{x} = \hat{A}\hat{A}^{(1,3)}\hat{b}$, since $\hat{A}(I - \hat{A}^{(1,3)}\hat{A}) = 0$. This completes the proof. \square

Remark 12. Note that the solution \hat{x} in Eq. (137) is not, in general, orthogonal to the null space of \hat{A} , which is $I - \hat{A}^+\hat{A}$ (see Corollary 4). For arbitrary \hat{w} , we have

$$\begin{aligned} (I - \hat{A}^+\hat{A})^T [\hat{A}^{(1,3)}\hat{b} + (I - \hat{A}^{(1,3)}\hat{A})\hat{w}] &= (I - \hat{A}^+\hat{A})\hat{A}^{(1,3)}\hat{b} + (I - \hat{A}^{(1,3)}\hat{A})\hat{w} \\ &= (\hat{A}^{(1,3)} - A^{(1,2,3,4)}\hat{A}\hat{A}^{(1,3)})\hat{b} + (I - \hat{A}^{(1,3)}\hat{A})\hat{w} \neq 0, \end{aligned} \tag{147}$$

in general. Note that $\hat{A}^{(1,3)} \neq A^{(1,2,3,4)}\hat{A}\hat{A}^{(1,3)}$, in general, since $\hat{A}^{(1,3)}$ does not necessarily satisfy the 2nd MP dual condition. \blacklozenge

Remark 13. The analogous result to Eq. (146) when the m -by- n matrix A and the vector b are real is the following. The inconsistent equation $Ax = b$ has the least squares solution $x = A^{(1,3)}b + (I - A^{(1,3)}A)w$, where w is an arbitrary n -by-1 column vector. In general, this solution, which is analogous to Eq. (137), is not unique. The solution vector x minimizes the Euclidean norm of the error $e := Ax - b$, and the unique minimum value of the Euclidean norm of the error is given (for all the solutions) by

$$\|e\|_{min} = \|(AA^{(1,3)} - I)b\|. \tag{148}$$

When (all) the dual parts of the linear dual equation $\hat{A}\hat{x} = \hat{b}$ are zero, $\langle \hat{e} \rangle$ reduces to that given in Eq. (148), furthering the correspondence between the result for the least-squares analog of a system of linear dual equations with that for a system of linear real equations. \blacklozenge

When the primal part of the m -by- n dual matrix \hat{A} has full row or full column rank, we know from Results 8 and 9 that a $\{1, 3\}$ -dual inverse matrix always exists, and also how to find one. When the primal part of the dual matrix \hat{A} has rank, $r \leq m, n$ then it does not generically speaking have a $\{1\}$ -dual inverse, and therefore also a $\{1,3\}$ -dual inverse. We focus on this non-generic situation next.

Result 13. When the primal part, A , of the dual matrix $\hat{A} = A + \varepsilon B$ has rank $r \leq m, n$, and satisfies the relation $U_2^T B V_2 = 0$, where U_2 and V_2 come from the singular value decomposition of A , and are defined in Eq. (103), then $\hat{A}^{\{1,3\}} = A^{\{1,3\}} + \varepsilon R$, where R is the solution, if it exists, to the linear set of equations

$$V_1^T R U_1 = -(\Lambda^{-1} P \Lambda^{-1} + \Lambda^{-1} Q L), \tag{149}$$

and

$$B A^{\{1,3\}} + A R - (B A^{\{1,3\}} + A R)^T = 0. \tag{150}$$

The matrices P and Q are defined in Eq. (109).

Proof: The singular value decomposition of the matrix $A = U \Lambda V^T$ in which U and V are orthogonal matrices can be used to get the $\{1,3\}$ -inverse of the matrix A , which always has the form [18,19]

$$A^{\{1,3\}} = V \begin{bmatrix} \underbrace{r \times r}_{\Lambda^{-1}} & \underbrace{r \times (m-r)}_0 \\ \underbrace{L}_{(n-r) \times r} & \underbrace{0}_{(n-r) \times (m-r)} \end{bmatrix} U^T. \tag{151}$$

This can be easily verified by checking that this matrix satisfies the 1st and 3rd Moore-Penrose conditions. Comparing with Eq. (105), we see that this entails that the matrix $K = 0$, and $M = 0$.

Setting $K = 0$ in Eq. (113), the first Moore-Penrose dual condition gives Eq. (149). For the 3rd dual Moore-Penrose dual condition to be satisfied, we require Eq. (150). Hence the result. \square

Example 7. Consider the matrix

$$\hat{A} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix} + \varepsilon \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 14 \end{bmatrix} := A + \varepsilon B \tag{152}$$

given Eq. (120). Our aim is to find $\hat{A}^{\{1,3\}} = A^{\{1,3\}} + \varepsilon R$.

Noting that every $\{1,3\}$ -dual generalized inverse must also be a $\{1\}$ -dual generalized inverse, for a $\{1\}$ -inverse to exist we require that $U_2^T B V_2 = 0$, so that if we take $b_i, i = 1, \dots, 8$ in Eq. (117) we must have $b_9 = 14$, as before in Example 5.

We begin by finding $A^{\{1,3\}}$. We know that it is given by Eq. (151) in which Λ is given in Eq. (118). Choosing the matrix $L = [1 \ 3]$, gives (see Remark 7)

$$A^{\{1,3\}} = \begin{bmatrix} 0.0620 & -0.2172 & -0.1553 \\ 1.0620 & -1.2172 & -0.1553 \\ -1.5192 & 2.3184 & 0.7992 \end{bmatrix}. \tag{153}$$

Solving the two sets of linear relations in Eqs. (149) and (150) to get R , we obtain

$$R = \begin{bmatrix} -p_1/3 + 2.075494 & -p_2/3 - 4.933306 & -p_3/3 - 1.413124 \\ -p_1/3 + 2.680196 & -p_2/3 - 3.049158 & -p_3/3 - 1.924518 \\ p_1 & p_2 & p_3 \end{bmatrix}, \tag{154}$$

where $p_i, i = 1, 2, 3$, are arbitrary real numbers. The set of $\{1,3\}$ -dual generalized inverses of \hat{A} are the matrices $\hat{A}^{\{1,3\}} = A^{\{1,3\}} + \varepsilon R$. The $\{1,3\}$ -dual generalized inverse of \hat{A} is not, in general, unique.

Taking $p_1 = 3, p_2 = 3$, and $p_3 = 2$, gives a specific $\{1,3\}$ -dual generalized inverse

$$\hat{A}^{\{1,3\}} = A^{\{1,3\}} + \varepsilon R, \text{ with } R = \begin{bmatrix} 1.0755 & -5.9331 & -2.0798 \\ 1.6802 & -4.0492 & -2.5912 \\ 3.0000 & 3.0000 & 2.0000 \end{bmatrix}, \tag{155}$$

where $\hat{A}^{\{1,3\}}$ is given in Eq. (153).

Taking $p_1 = 1, p_2 = 2$, and $p_3 = 3$, gives another specific $\{1,3\}$ -dual generalized inverse (see Remark 7)

$$\hat{A}^{\{1,3\}} = A^{\{1,3\}} + \varepsilon R, \text{ with } R = \begin{bmatrix} 1.0755 & -5.9331 & -2.0798 \\ 1.6802 & -4.0492 & -2.5912 \\ 3.0000 & 3.0000 & 2.0000 \end{bmatrix} \tag{156}$$

Note that these $\{1,3\}$ -dual inverses do not satisfy the 2nd and the 4th MP dual conditions. \aleph

Example 8. Consider the dual matrix \hat{A} given in Eq. (152) and the inconsistent equation $\hat{A}\hat{x} = \hat{b}$, where

$$\hat{b} = [8.2, 7.3, 15.1]^T + \varepsilon [30.2, 32.8, 53.6]^T \tag{157}$$

Using $\hat{A}^{(1,3)}$ given in Eq. (155) in Eq. (137), we get the two right-hand members of the solution vector \hat{x} as (see Remark 7)

$$A^{(1,3)}\hat{b} = \begin{bmatrix} -3.4224 - \varepsilon 79.4739 \\ -2.5224 - \varepsilon 71.0849 \\ 16.5338 + \varepsilon 149.6970 \end{bmatrix} \tag{158}$$

and

$$(I - \hat{A}^{(1,3)}\hat{A})\hat{w} = \left\{ \begin{bmatrix} 1.8383 & 0.5591 & 0.4658 \\ 1.8383 & 0.5591 & 0.4658 \\ -5.5150 & -1.6773 & -1.3974 \end{bmatrix} + \varepsilon \begin{bmatrix} 17.8683 & 11.7914 & 12.4951 \\ 16.0300 & 11.2323 & 12.0293 \\ -20.5150 & -25.3099 & -29.1006 \end{bmatrix} \right\} \hat{w}, \tag{159}$$

where w is an arbitrary 3-by-1 dual vector. As indicated in Eq. (137), \hat{x} is the sum of these two components. We see clearly that the solution $\hat{x} = \hat{A}^{(1,3)}\hat{b} + (I - \hat{A}^{(1,3)}\hat{A})\hat{w}$ is not unique though the error norm

$$\langle \hat{e} \rangle = \langle (I - \hat{A}^{(1,3)}\hat{A})\hat{b} \rangle = \left\langle \begin{bmatrix} 0.1333 - \varepsilon 0.0778 \\ 0.1333 + \varepsilon 0.3222 \\ -0.1333 - \varepsilon 0.1889 \end{bmatrix} \right\rangle = 0.2309 + 0.3815 = 0.6124 \tag{160}$$

is. That is, every matrix $\hat{A}^{(1,3)} = A^{(1,3)} + \varepsilon R$ (Eqs. (153) and (154) for arbitrary values of the parameters $p_i, i = 1, 2, 3$ will yield the same value of $\langle \hat{e} \rangle$ given in Eq. (160).

Note that the solution $\hat{x} = \hat{A}^{(1,3)}\hat{b} + (I - \hat{A}^{(1,3)}\hat{A})\hat{w}$ is not, in general, orthogonal to the null space of \hat{A} (see Remark 12) since (with $\hat{w} = 0$)

$$(I - \hat{A}^+\hat{A})^T \hat{A}^{(1,3)}\hat{b} = \begin{bmatrix} -5.0496 \\ -5.0496 \\ 15.1489 \end{bmatrix} + \varepsilon \begin{bmatrix} -77.7255 \\ -72.6758 \\ 142.2828 \end{bmatrix} \neq 0. \tag{161}$$

⌘

3.3. Analogue of the least-squares minimum-norm solution of the possibly inconsistent dual equation $\hat{A}\hat{x} = \hat{b}$

We consider here the dual analogue of the least-squares minimum norm solution of the equation $Ax = b$ where A and b are real. The Moore-Penrose inverse is often used in inverse problems in various areas of application. Ref. [17], for example, shows an application from the area of kinematics in which noise-corrupted measurements (or simulated data) made both before and after the rotation of a solar panel are used to solve the inverse problem of estimating the dual rotation matrix. In the interest of brevity this example is not presented here. The interested reader can look at this somewhat interesting example, which is worked out in considerable detail, in Ref. [17].

Result 14. Consider the possibly inconsistent equation $\hat{A}\hat{x} = \hat{b}$ in which $\hat{A} = A + \varepsilon B$ is an m -by- n dual matrix. If the Moore-Penrose dual generalized inverse (MPDGI) of the matrix \hat{A} exists, the dual vector $\hat{x} = p + \varepsilon q$ for which the norm of the error $\langle e \rangle = \langle Ax - b \rangle$ is given by

$$\langle e \rangle = \langle (AA^{(1,3)} - I)\hat{b} \rangle \tag{162}$$

has the unique solution vector \hat{x} given by

$$\hat{x} = \hat{A}^+\hat{b}, \tag{163}$$

which is orthogonal to the null space of \hat{A} .

Proof: Since the $\{1,2,3,4\}$ -dual inverse, $\hat{A}^{\{1,2,3,4\}}$, satisfies all the four MP dual conditions (see Eqs. (8)–(11), it obviously satisfies the first and third condition. Hence it is also a $\{1,3\}$ -dual generalized inverse of \hat{A} .

Consider

$$\hat{x} = \hat{A}^{\{1,2,3,4\}}\hat{b} + (I - \hat{A}^{\{1,2,3,4\}}\hat{A})\hat{w} = \underbrace{\hat{A}^+\hat{b}}_{\hat{v}_1} + \underbrace{(I - \hat{A}^+\hat{A})\hat{w}}_{\hat{v}_2}. \tag{164}$$

Clearly,

$$\hat{A}\hat{x} = \hat{A}\hat{A}^{\{1,2,3,4\}}\hat{b} \tag{165}$$

since $\hat{A}(I - \hat{A}^{\{1,2,3,4\}}\hat{A}) = 0$ by the 1st MP dual condition (Eq. (8)). As, mentioned, $\hat{A}^{\{1,2,3,4\}}$ is a $\{1,3\}$ -dual generalized inverse of \hat{A} and since Eq. (165) is true, from Result 12 we see that Eq. (162) is true.

The inner product

$$\hat{v}_2^T \hat{v}_1 = \hat{w}^T (I - \hat{A}^+ \hat{A})^T \hat{A}^+ \hat{b} = \hat{w}^T (I - \hat{A}^+ \hat{A}) \hat{A}^+ \hat{b} = 0. \tag{166}$$

In the second equality above we have used the fact that \hat{A}^+ satisfies the fourth MP condition, and in the last equality that it satisfies the second MP dual condition, which is $\hat{A}^+ - \hat{A}^+ \hat{A} \hat{A}^+ = 0$.

We then find that

$$\|\hat{x}\|^2 = \|v_1\|^2 + \|v_2\|^2 = \left\| \underbrace{\hat{A}^+ \hat{b}}_{p_1 + \varepsilon q_1} \right\|^2 + \left\| \underbrace{(I - \hat{A}^+ \hat{A}) \hat{w}}_{p_2 + \varepsilon q_2} \right\|^2. \tag{167}$$

Denoting the dual vectors $\hat{v}_i = p_i + \varepsilon q_i, i = 1, 2$, we obtain, as before, a bound on the norm of the dual vector \hat{x} given by

$$\langle \hat{x} \rangle \leq \langle \hat{x} \rangle_{bound} = \sqrt{\|p_1\|^2 + \|p_2\|^2} + \|q_1\| + \|q_2\|. \tag{168}$$

To make $\langle \hat{x} \rangle_{bound}$ as small as possible for all vectors \hat{b} , we make the vector $\hat{v}_2 = 0$, so that $\|p_2\| = \|q_2\| = 0$, by setting $w = 0$. From Eq. (167) we then get

$$\|\hat{x}\|^2 = \|v_1\|^2 = \|\hat{A}^+ \hat{b}\|^2, \tag{169}$$

so that

$$\langle x \rangle = \langle v_1 \rangle = \|p_1\| + \|q_1\| = \langle \hat{A}^+ \hat{b} \rangle. \tag{170}$$

By Corollary 4, the null space of \hat{A} is $(I - \hat{A}^+ \hat{A})$, and the vector $\hat{x} = \hat{A}^+ \hat{b}$ is orthogonal to this null space since $(I - \hat{A}^+ \hat{A})^T \hat{A}^+ = [I - (\hat{A}^+ \hat{A})^T] \hat{A}^+ = [I - \hat{A}^+ \hat{A}] \hat{A}^+ = 0$.

Note that the error $\hat{e} = A\hat{x} - \hat{b}$ in satisfaction of the equation $A\hat{x} = \hat{b}$ is the same as that given in Eq. (162) since the (unique) {1,2,3,4}-dual generalized inverse is also a {1,3}-dual generalized inverse. \square

Example 9. We consider again the matrix \hat{A} given by Eq. (152) and the inconsistent equation $\hat{A}\hat{x} = \hat{b}$ with \hat{b} given again by Eq. (157) from Example 8. MPDGI is given by the matrix is $\hat{A}^+ = A^+ + \varepsilon R$, where (see Remark 7)

$$A^+ = \begin{bmatrix} -0.4545 & 0.5455 & 0.0909 \\ 0.5455 & -0.4545 & 0.0909 \\ 0.0303 & 0.0303 & 0.0606 \end{bmatrix}. \tag{171}$$

Solving Eqs. (34)–(37) for the matrix R , we get

$$R = \begin{bmatrix} -0.9394 & -0.4848 & 0.0303 \\ 0.1818 & 0.6364 & -0.7273 \\ -0.2525 & 0.3838 & 0.1010 \end{bmatrix}. \tag{172}$$

The solution \hat{x} to these inconsistent dual equations is the given by

$$\hat{x} = \underbrace{\begin{bmatrix} 1.6273 \\ 2.5273 \\ 1.3848 \end{bmatrix}}_{p_1} + \varepsilon \underbrace{\begin{bmatrix} -1.7485 \\ 1.5909 \\ 7.4141 \end{bmatrix}}_{q_1} \tag{173}$$

whose norm is $\langle \hat{x} \rangle = \|p_1\| + \|q_1\| = 11.09$. The error $\hat{e} = A\hat{x} - \hat{b}$ is

$$\hat{e} = \begin{bmatrix} 0.1333 - \varepsilon 0.0778 \\ 0.1333 + \varepsilon 0.3222 \\ -0.1333 - \varepsilon 0.1889 \end{bmatrix}$$

whose norm is 0.6124. It has the same norm as that given in Eq. (160), since \hat{A}^+ is also a {1,3}-dual generalized inverse of the matrix \hat{A} , and every {1,3}-dual generalized inverse, $\hat{A}^{\{1,3\}}$, yields the same error norm (Corollary 7). \aleph

Remark 14. The analogous result when the m -by- n matrix A and the vector b are real is the following. The inconsistent equation $Ax = b$ has the unique least squares solution $x = A^+b$ which is orthogonal to the null space of A , and has minimum Euclidean norm. When the dual parts of the dual equation are zero, the results given in Eqs. (162) and (163) coincide with those when A and b are real. \blacklozenge

4. Conclusions

The first part of this paper provides new insights into, and properties of, dual generalized inverses of dual matrices. It starts with the development of a suitable norm of a dual vector and its connection with the Euclidean norm of the dual vector, which is defined as the inner product of the dual vector with itself. After defining various types of dual generalized inverses, the properties of these inverses are provided. The second part of this paper explores solutions of linear dual equations which are commonly encountered in several areas of kinematics and dynamics of machines and mechanisms. The analytical results are illustrated using extensive numerical computations. Since the paper is some-what long we summarize below some of the salient results obtained.

Some of the new results from the first part of the paper are as follows:

1. Results on dual matrices and their generalized inverses under orthogonal and non-orthogonal linear transformations.
2. If a Moore-Penrose dual generalized inverse (MPDGI) of a dual matrix exists, it is unique.
3. Several properties of the MPDGI (when it exists) of a dual matrix are given in Result 6.
4. An explicit formula for the MPDGI of any matrix, if it exists, whose primal part has full row(column) rank is obtained directly in terms of its primal and dual parts. The formula is compared with known computational algorithms that use singular value decompositions.
5. When \hat{A}^+ exists, the matrices $\hat{A}\hat{A}^+$, $\hat{A}^+\hat{A}$, $I - \hat{A}\hat{A}^+$, and $I - \hat{A}^+\hat{A}$ are idempotent, indicating that these matrices are geometrically dual projection operators.

Some of the new results provided in the second part of the paper, which addresses the dual equation $\hat{A}\hat{x} = \hat{b}$ that commonly arises in areas like kinematics and robotics, are as follows.

1. The necessary and sufficient condition for a system of linear dual equations to have a solution under the assumption of the existence of a {1}-dual generalized inverse of \hat{A} .
2. The explicit solution to $\hat{A}\hat{x} = \hat{b}$ in terms of the {1}-dual generalized inverse of \hat{A} . The solution is shown, in general, to be non-unique.
3. The explicit solution of the homogenous equation $\hat{A}\hat{x} = 0$, giving the null space of \hat{A} in terms of the {1}-dual inverse of \hat{A} .
4. The necessary and sufficient conditions for a dual matrix \hat{A} to have a {1}-dual generalized inverse. The {1}-dual generalized inverse of \hat{A} , in general, is non-unique.
If these conditions are not satisfied by every {1}-dual inverse, the dual matrix \hat{A} does not have a {1,3}-dual generalized inverse nor a Moore-Penrose dual generalized inverse.
5. When the primal part of \hat{A} is rank deficient, its primal and dual parts cannot be independently chosen for it to have a {1}-dual generalized inverse; such dual matrices are non-generic and most likely have no Moore-Penrose dual inverse and no dual analog of a least-squares generalized inverse.
6. An explicit expression to obtain a {1}-dual generalized inverse, when it exists, of \hat{A} . Generation of infinitely many {1}-dual generalized inverses, in general, given a {1}-dual generalized inverse.
7. Analogue of the least-squares solution of the possibly inconsistent dual equation $\hat{A}\hat{x} \approx \hat{b}$. The dual solutions are shown to be non-unique, in general. When all the dual parts of the dual equation are zero and the equation becomes real, and the dual solutions reduce to the well-known least-squares solutions for real equations, whose properties the dual solution then inherits.
8. Analogue of the least-squares minimum norm solution for real matrices when we have a system of linear dual equations. When all the dual parts of the dual equations are zero and the equations becomes real, the dual solutions reduce to the well-known least-squares minimum-norm solutions for the real equations.

Declaration of Competing Interest

There is no conflict of interest in this paper.

References

- [1] D. de Falco, E. Pennestrì, F.E. Udwadia, On generalized inverses of dual matrices, *Mech. Mach. Theory* 123 (2018) 89–106 -05.
- [2] McCarthy A. Perez-Gracia, Bennett's linkage and the cylindroid, *Mech. Mach. Theory* 33 (2002) 1245–1260.
- [3] E. Pennestrì, P.P. Valentini, G. Figliolini, J. Angeles, Dual Cayley-Klein parameters and Möbius transform: theory and applications, *Mech. Mach. Theory* 106 (2016) 50–67.
- [4] A. Cohen, M. Shoham, Application of hyper-dual numbers to multibody kinematics, *ASME J. Mech. Robot.* 8 (2016) 011015.
- [5] J. Yang, X. Wang, The application of dual-numbers to scara kinematics, *Intl. Conf. Mech. Autom. Control Eng.* (2010).
- [6] M.L. Keler, in: *Dual Vectors of the Dynamics and Kinematic Kind in Matrices, Recent Advances in Robot Kinematics*, Kluwer Academic Publishers, 1996, pp. 245–254.
- [7] H. Heiss, Homogeneous and dual matrices for treating the kinematic problem of robots, *IFAC Proc.* 19 (4) (1986) 51–54.
- [8] Gu You-Liang, J.Y.S. Luh, Dual-number transformations and its applications to robotics, *IEEE J. Robot. Autom.* RA-3 (615623) (1987).
- [9] M.L. Keler, Dual vectors of the dynamics and kinematic kind in matrices, in: *Recent Advances in Robot Kinematics*, Kluwer Academic Publishers, 1996, pp. 245–254.
- [10] V. Brodsky, M. Shoham, Derivation of dual forces in robot manipulators, *Mech. Mach. Theory* 33 (1998) 1241–1248.

- [11] E. Pennestri, P.P. Valentini, Linear dual algebra algorithms and their application to kinematics, multibody dynamics, *Comput. Methods Appl. Sci.* 12 (2009) 207–229.
- [12] E. Pennestri, P.P. Valentini, D. de Falco, The Moore{penrose dual generalized inverse matrix with application to kinematic synthesis of spatial linkages, *ASME J. Mech. Des.* 140 (2018) 102303 1–102303-7.
- [13] B. Belzile, J. Angeles, Reflections over the dual ring – applications to kinematic synthesis, *ASME J. Mech. Des.* 141 (7) (2019) 072302 1-072302.9.
- [14] A. Magarena, J. Santolaria, J. Aguilar, An overview of kinematic and calibration models for internal/external sensors or constraints to improve the behavior of spatial parallel mechanisms, *Sensors* 10 (11) (2010) 10256–10297.
- [15] A. Jobair, I. Boney, Kinematic calibration of a 6-axis serial robot using distance and sphere constraints, *Int. J. Adv. Manuf. Technol.* 77 (2015) 515–523.
- [16] F.E. Udwadia, E. Pennestri, D. de Falco, Do all dual matrices have dual moore-penrose generalized inverses? *Mech. Mach. Theory* 151 (2020) 103878.
- [17] C.R. Rao, S.K. Mitra, *Generalized Inverses and its Applications*, John Wiley, 1972.
- [18] F.E. Udwadia, *Analytical Dynamics: a New Approach*, second ed., Cambridge University Press, 2007.